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TOPICS IN NONCOMPACT GROUP REPRESENTATIONS

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INTRODUCTION

The theory of Lie algebras, Lie groups and their representations is a subject which hardly needs of a lengthy intoduction to highlight its intrinsec mathematical interest and manifold applications in physics. It is also a subject whose exposition in a course for physicists may be done from a wide variety of starting points, directions, speeds and objectives. We assume that the audience of physicists is reasonably confident in the use of quantum mechanics, angular momentum theory and the standard SU(3) methods of the old quark model, so we shall proceed presenting some of the Lie algebraic and group theoretical concepts, offering alternatives and generalizations, with the aim that the subject acquire a certain systematic structure. This should enable the student to work more at ease when applying the methods of Lie algebras and groups to the problems in his field.

The territory we will try to make more accessible is that of representations of noncompact algebras and groups. Symmetry algebras and groups à la Wigner-Raccah are probably standard in any physics curriculum, but dynamical algebras and the associated groups are less so. The similarities between the two are sufficient to use the concepts of the former in order to gain familiarity with the latter; but the differences in formalism and applications are sometimes dramatic. Unitary representations of noncompact groups are infinite dimensional, and generally describe covariance instead of invariance of a system.

Chapter 1 deals with the Heisenberg-Weyl algebra of quantum mechanics, and Chapter 2 with the corresponding group. This is an easy case, as the group is nilpotent and close to the abelian case. Chapter 3 explores a semisimple noncompact case, the 2+1 Lorentz algebra and group, as well as its covering groups.

We do not claim to present everything you always wanted to know in this field, as have other authors in References (1) and (2), so we hardly need to start with a disclaimer. We do wish to point out, however, that many of the prime areas of research in the last years have been left out, notably higher dimensional cases and their classification, similarity methods in differential equations, induced representations, pseudogroups, superalgebras and gauge theories.

It is hoped that this material will prove useful for mathematically inclined physicists. We mut apologize both to the purer mathematical analysts for glossing over defining concepts or presenting proofs; also to the applied physicists, for not presenting the energy levels' match of any molecule, atom, nucleus or resonance. We hope our list of references will partially atone for the sins of omission. Those of comission, we also hope, will entice the reader to explore the source literature.

CHAPTER 1: A LIE ALGEBRA

We start this set of lecture notes with an example -a rather simple and presumably well-know one- of a Lie algebra. We shall then examine its representations, its irreducible representations, and finally, its unitary irreducible representations.

1.1 A Lie algebra and its bases.

We consider the Lie algebra w with elements Q, P and I, over the field of complex numbers, defined by its Lie bracket operation

$$[Q,P]=iI,\quad [Q,I]=0,\quad [P,I]=0. \tag{1.1}$$

The symbols used for Q and P suggest, of course, that these are the quantum-mechanical operators of position (multiplication by the coordinate q), momentum ($-i\hbar$ times differentiation with respect to q) while I should be a multiple $\hbar$ of the unit operator 1, on the space of quantum mechanical wavefunction completed with respect to an inner product so as to form a Hilbert space, normally $L^2(R)$. We shall later obtain these as a realization of w, but meanwhile they are to be taken only as formal symbols.

The three elements Q, P and I constitute a *vector* basis for the algebra, in the sense that any element of w can be written as

$$E = xQ + yP + zI \quad , x,y,z \in C \tag{1.2}$$

An *algebraic* basis for w is provided by Q and P alone, as I is produced through the Lie bracket in the first of Eqs. (1.1).

Particular elements in w which we will refer to are

$$R = \frac{1}{\sqrt{2}}(Q-iP),\qquad L = \frac{1}{\sqrt{2}}(Q+iP). \tag{1.3a}$$

These are to be related, later, with the raising and lowering operators for the harmonic oscillator wavefunctions. They satisfy

$$[L,R] = I\ ,\quad [L,I] = 0,\ [R,I] = 0, \tag{1.3b}$$

and may thus also serve to define w over the field C.

1.2 Representations.

A *representation* ρ of a Lie algebra a on a vector space V is a *homomorphism* (i. e. a mapping which preserves the operators of the algebra: Linear combination and Lie brackets)

$$\rho : a \longrightarrow gl(V)\ , \tag{1.4}$$

from $\mathcal{a}$ *into* the algebra gl(V) of linear operators on V, where the Lie bracket is the commutator.

If $\{X_j\}_{j=1}^{D}$ is a D-dimensional vector basis for $\mathcal{a}$, abstractly defined through the set of structure constants c_{jk}^{ℓ} as

$$[X_j, X_k] = i \sum_{\ell=1}^{D} c_{jk}^{\ell} \, x_\ell , \tag{1.5}$$

and $\mathsf{X}_j = \rho(x_j) \in$ gl(V), then the homomorphism requirements are

$$\rho(c_j X_j + c_k X_k) = c_j \, \rho(X_j) + c_k \, \rho(X_k) = c_j \mathsf{X}_j + c_k \mathsf{X}_k \tag{1.6a}$$

$$\rho([X_j, X_k]) = \mathsf{X}_j \mathsf{X}_k - \mathsf{X}_k \mathsf{X}_j = i \sum_{\ell=1}^{D} c_{jk}^{\ell} \, \mathsf{X}_\ell . \tag{1.6b}$$

When V is an N-dimensional vector space, gl(V) is the set of $N \times N$ matrices. Amongst these, for w, we should be able to find three matrices Q, P and I such that (1.1) holds replacing each symbol for its boldface homonym, and the bracket meaning commutation. Abstractly, you will recall that the Lie bracket is only required to be skew-symmetric $[A,B] = -[B,A]$, bilinear $[aA+bB,C] = a[A,C] + b[B,C]$, and to satisfy the Jacobi identity $[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$. Poisson (3, Sect III-D and V-A) and Moyal (4,5) brackets, as well as commutators, satisfy these three requirements.

When V is a function- or other infinite-dimensional space, gl(V) is the set of all linear operators with domain and range in that space. Amongst these we should be able to find, for w, three linear operators $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{I}$ such that (1.1) holds for them. Clearly here we run headlong into trouble, since we can easily find that some proposed operator does not '*quite*' have V for its domain, or that it may send elements of V out of V. We may require that it do the job only in a dense subspace of V, and that will take us into Hilbert spaces.

1.3 The adjoint representation.

For any D-dimensional Lie algebra $\mathcal{a}$ [with a D-dimensional vector basis $\{X_j\}_{j=1}^{D}$, defined by the structure constants $c_{jk}^{\ell} = -c_{kj}^{\ell}$ in the Lie Bracket (1.5)] one can always produce a $D \times D$ matrix representation, called the *adjoint* representation ρ^A of $\mathcal{a}$, through

$$\mathsf{X}_k^A = \rho^A(X_k), \qquad (\mathsf{X}_k^A)_{mn} = i \, c_{m\,k}^{n} = -i \, c_{k\,m}^{n} . \tag{1.7a}$$

To prove this, replace (1.7a) into (1.5), X_k's replacing X_k's, thus finding for the m-n element of this equality

$$\sum_s [c_{mj}^{s} c_{ks}^{n} + c_{km}^{s} c_{js}^{n} + c_{jk}^{s} c_{ms}^{n}] = 0, \tag{1.7b}$$

after a dummy-index change. Equation (1.6) is equivalent to the statement of the Jacobi identity for the algebra.

The adjoint representation of w may be obtained from (1.7), numbering Q, P and I in (1.1) by 1, 2, 3. As only $c^3_{21} = -1$ and $c^3_{12} = 1$ are nonzero,

$$Q^A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{pmatrix}, \quad P^A = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I^A = 0 . \tag{1.8}$$

This is a representation, but it is not *faithful*, as the algebra element I is mapped on the zero matrix. This is a general feature of algebras with *a centre*, i. e. element(s) which commute with every element of the algebra.

1.4 A faithful 3×3 matrix representation.

For the case of w we have to use other arguments: If Q is represented by a matrix with a single off-diagonal nonzero element in the (a,b) position, $a \neq b$, and P by another such matrix with a nonzero (c,a) element, $c \neq a$, the commutator representing I will have non-zero elements in the (b,c) position and, if $b=c$, in the (a,a) position. The latter will commute with the former two when the (a,a) element is zero, i. e., when $b \neq c$. Adjusting signs and i' s so that (1.1) be obtained, we may thus provide a simple 3×3 faithful representation through $a=1$, $b = 2$, $c = 3$:

$$Q^\delta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P^\delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad I^\delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}. \tag{1.9a}$$

As a representation is a linear mapping, every element (1.2) of w may be represented through

$$E^\delta = x\,Q^\delta + y\,P^\delta + z\,I^\delta = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ y & iz & 0 \end{pmatrix}. \tag{1.9b}$$

1.5 Equivalent representations and isomorphisms of w.

We may perform any similarity transformation on representation matrices (as $M^\delta \to M^{\delta e} = A\, M^\delta A^{-1}$ for $M^\delta = Q^\delta$, P^δ and I^δ) and obtain another *equivalent* representation. For w we may do more, however: We can perform linear combinations of its vector basis elements obtaining new basis elements with the *same* commutation relations:

$$\begin{pmatrix} \bar{Q} \\ \bar{P} \\ \bar{I} \end{pmatrix} = \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & ad-bc \end{pmatrix} \begin{pmatrix} Q \\ P \\ I \end{pmatrix}, \qquad a,\ b,\ c,\ d,\ u,\ v, \in C , \tag{1.10}$$

i. e. (1.1) for Q, P and I implies (1.1) also holds for Q, P and

I implies (1.1) also holds for $\mathcal{Q}$, $\mathcal{P}$ and $\mathcal{I}$. The linear transformation (1.10) is an isomorphism of $\mathcal{w}$. This is a rather exceptional case, as usually the isomorphism group of a Lie algebra is only the adjoint action of the Lie group associated to it. For $\mathcal{w}$ it is $I^2GL(2,C)$.

The above paragraph binds the 'simplest' representation (1.9) to the more popular representation [3, Eq' (2.4); 6, Eqs. (2.1)] given by

$$Q^p = \begin{pmatrix} 0 & -i & -i \\ i & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad P^p = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad I^p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{pmatrix}. \tag{1.11}$$

It is more popular since it may be generalized straighforwardly as an (N+2) × (N+2) representation of the (2N+1)-dimensional Heisenberg-Weyl algebra used in N-dimensional quantum mechanics, through 'vectorizing' the nonzero elements of Q^p and P^p. As matrices, the link between (1.11) and (1.9) is

$$A^{-1}\; Q^p\; A = \alpha\, Q^\sigma + \beta\, P^\sigma + \gamma I^\sigma \tag{1.12a}$$

$$A^{-1}\; P^p\; A = -i\alpha\, Q^\sigma + i\beta P^\sigma + i\gamma I^\sigma \tag{1.12b}$$

$$A^{-1}\; I^p\; A = -2i\alpha\beta I^\sigma \tag{1.12c}$$

where in this case

$$A = \begin{pmatrix} \beta & i\gamma & 0 \\ 0 & \alpha\beta & 1 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.12d}$$

1.6 The Fock realization.

The Heisenberg-Weyl algebra can be also furnished with infinite-dimensional representations. To this end we prefer to work with the basis R, L and I as given by (1.3). We may propose to associate to each of these elements the formal Fock operators

$$R \longrightarrow R^F = z, \quad L \longrightarrow L^F = \frac{d}{dz}, \quad I \longrightarrow I^F = I, \tag{1.13}$$

i. e. L^F is the differentiation operator, $R^F f(z) = z f(z)$ is the multiplication-by-argument operator, and I^F, the unit operator. Acting on any differentiable function $f(z)$ we can see that (1.13) follow the commutation relations (1.3b). We prefer to speak of (1.13) as a *realization* of the algebra $\mathcal{w}$, since a representation is defined only when a proper vector space V in (1.4) is fully specified. The Fock operators are purely formal up to now, with no mention as to their domain.

1.7 The power-function basis.

Clearly, we need some functions. Consider the infinite sequence of *power* functions

$$p_n^{\nu}(z) = z^{\nu+n}, \quad n \in Z = \{0,\pm 1,\pm 2,\ldots\}, \mathrm{Re}\ \nu \in [0,1). \tag{1.14}$$

These provide a proper basis for a function space V as they are differentiable, and (1.13) belong to gl(V):

$$\mathrm{R}^F\, p_n^{\nu}(z) = p_{n+1}^{\nu}(z), \tag{1.15a}$$

$$\mathrm{L}^F\, p_n^{\nu}(z) = (n+\nu)\, p_{n-1}^{\nu}(z), \tag{1.15b}$$

$$\mathrm{I}^F\, p_n^{\nu}(z) = p_n^{\nu}(z). \tag{1.15c}$$

The action of (1.13) can be extended to the whole of V through linear combiation. Note carefully that the vector space axioms speak only of finite linear combinations, and hence V is *not* the space of z^{ν} times analytic functions in some annullus around the origin (having a convergent Laurent expansion), but only the subspace of finite sums of power functions. The limit points are the *closure* of that space, of which we shall have a right to talking only when we introduce a *norm* into that space. Meanwhile, though, we do have a representation through infinite matrices with rows and columns $n,m \in Z$, the set of integers:

$$\mathrm{R}^{F(\nu)} = \begin{array}{c|ccccc|c} n\backslash m & 2 & 1 & 0 & -1 & -2 & \cdots \\ \hline 2 & 0 & 1 & & & & \\ 1 & & 0 & 1 & & & \\ 0 & & & 0 & 1 & & \\ -1 & & & & 0 & 1 & \\ -2 & & & & & 0 & \\ \vdots & & & & & & \ddots \end{array} \tag{1.16a}$$

$$
L^{F(\nu)} \qquad
\begin{array}{c}
n\backslash m \\ \downarrow \\ 2 \\ \downarrow \\ 1 \\ \downarrow \\ 0 \\ \downarrow \\ -1 \\ \downarrow \\ -2 \\ \downarrow \\ \vdots
\end{array}
\begin{array}{c}
\rightarrow 2 \rightarrow 1 \rightarrow 0 \rightarrow -1 \rightarrow -2 \rightarrow \cdots \\
\left(\begin{array}{ccc|ccc}
0 & & & & & \\
\nu+2 & 0 & & & & \\
\hline
& \nu+1 & 0 & & & \\
& & \nu & 0 & & \\
& & & \nu+1 & 0 & \\
& & & & & \ddots
\end{array}\right)
\end{array}
\tag{1.16b}
$$

$$
I^{F(\nu)} = 1, \tag{1.16c}
$$

acting on infinite-component vectors, where $p_n^{\nu}(z)$ is represented by a column with a single nonzero entry 1 in the n^{th} position. This entry is raised by one place by the raising matrix (1.16a), lowered and multiplied by $n + \nu$ by the lowering matrix (1.16b), and left invariant by (1.16c).

Clearly also $R^{F(\nu)}$, $L^{F(\nu)}$ and $I^{F(\nu)}$ given by (1.16) satisfy the commutation relations (1.3b): $L^{F(\nu)}R^{F(\nu)}$ and $R^{F(\nu)}L^{F(\nu)}$ are diagonal matrices with entries $\nu + n + 1$ and $\nu + n$ along the diagonal.

1.8 Irreducible and indecomposable representations.

When ν is not zero, every $p_n^{\nu}(z)$ may be moved and, through repeated application of $R^{F(\nu)}$ or $L^{F(\nu)}$, taken to a function proportional to any other $p_m^{\nu}(z)$, $m \in Z$. No subspace being invariant, under the three algebra generators; the whole space spanned by $\{p_n(z)\}_{n\in Z}$ is thus required as a basis for V, which is thus an *irreducible* representation basis.

When $\nu=0$ then $L^{F(\nu)}$ has a zero in the 0^{th} column, and something dramatic happens: If $n < 0$, $p_n^0(z)$ may be lowered or raised to any other $p_m^0(z)$, $m \in Z$, but if $n \geqslant 0$ it may be raised, but not lowered below $n=0$. The space spanned by $\{p_n^0(z), n \in Z\}$ may be divided thus into two disjoint subbases: $\{p_n^0(z), n \geqslant 0\}$ which still transforms irreducibly under the algebra, and the subspace generated by $\{p_n^0(z), n < 0\}$ which does not: It '*spills over*' into the first through repeated action or the raising operator. Every element of w in the Fock representation (1.16) for $\nu = 0$ and its powers, are represented by a block-triangular infinite matrix acting on the basis vectors as

$$\left(\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right)\left(\begin{array}{c} P^0_{(\geqslant 0)}(z) \\ \hline P^0_{(<0)}(z) \end{array}\right) = \left(\begin{array}{c} A\,P^0_{(\geqslant 0)}(z) + B\,P^0_{(<0)}(z) \\ \hline C\,P^0_{(<0)}(z) \end{array}\right) \qquad (1.17)$$

where we have separated by horizontal and vertical lines on 0^{th} and $(-1)^{th}$ rows and columns. The representation of w provided by $\mathbb{A}$ is still irreducible, but (1.16) is reducible, though *indecomposable* since, as can be verified, it can *not* be decomposed fully into a block-diagonal form signifying two (or more) irreducible parts. Triangular matrices cannot be diagonalized.

1.9 Self-adjoint representations.

In Lie group theory one is generally interested in hermitian or self-adjoint irreducible representations of Lie algebras (1.5); thence the apparently unconfortable i in (1.5)-(1.6b), which insures that we may revert to the algebra a over the real field and still retain hermiticity for every element of the algebra. For our Lie algebra w, this is needed by the current axioms of Quantum Mechanics as given by Dirac and von Neumann (7, 8 Sect. 2.1, 9) which require that the position and momentum operators $\mathbb{Q}$ and $\mathbb{P}$ representing Q and P in w be self-adjoint -or rather, have self-adjoint extensions- in $L^2(R)$. This turns into a statement of intention to reduce our attention to representations (1.4) where V is a *Hilbert* space endowed with and Cauchy-sequence closed under an appropiate positive definite inner product $(\cdot,\cdot): V\times V \to C$. Axioms require that it be linear in one argument (physicists prefer the second argument) antilinear in the first, $(\tilde{f},\tilde{f}) \geqslant 0$ and $(\tilde{f},\tilde{f}) = 0$ iff $\tilde{f} = \tilde{0}$. For some purposes a *norm* rather than an inner product is needed, so V is taken to be only a Banach space, but we shall forego this weaker scenario.

1.10 Hermiticity imposed.

Returning to our now self-adjoint representation for Q, P and I in w, the reader will no doubt recognize the Schrödinger realization as possessing a self-adjoint extension in $L^2(R)$. In accord with our emphasis on alternate approaches, however, let us insist in staying with the Fock realization (1.13) which from (1.3a) must be such that $\mathbb{L}^{F\dagger} = \mathbb{R}^F$, $\mathbb{R}^{F\dagger} = \mathbb{L}^F$ and of course $\mathbb{I}^{F\dagger} = \mathbb{I}^F$. What we need is a Hilbert space B such that these adjunction properties hold, i. e., an inner product $(\cdot,\cdot)_B$ where the formal Fock operators (1.13) satisfy

$$(\mathbb{L}^F \tilde{f}, \tilde{g})_B = (\tilde{f}, \mathbb{R}^F \tilde{g})_B \quad \text{for all } \tilde{f}, \tilde{g}, \in B, \qquad (1.18)$$

plus some function-analytic minutae to determine the geometry of the space of functions which belong to $\mathcal{B}$ so that the limit points of Cauchy sequences of these are in the space.

It has been the work of Bargmann (10) to turn (1.18) into a set of two coupled partial differential equations for the weight function of a measure on the complex plane, and to determine precisely the geometry of the space -thereafter called *Bargmann's* space. Generalizations of this procedure have been performed by Barut and Girardello (11) and the author (12,13,14 Sect. 9.2). Let us follow a different argumentation here, based on the matrix representation (1.16) which we have already, and which we want to turn into a hermitian matrix representation.

An orthonormal denumerable basis for the putative Hilbert space $\mathcal{B}$, $\{g_n^\nu(z),\ n \in ?\}$ should have the property of providing a matrix representation of w through

$$(\mathbb{L}^{F'(\nu)})_{mn} = (\underset{\sim}{q}_m^\nu, \mathbb{L}^F \underset{\sim}{q}_n^\nu)_{\mathcal{B}} = (\mathbb{R}^F \underset{\sim}{q}_m^\nu, \underset{\sim}{q}_n^\nu)_{\mathcal{B}} = (\underset{\sim}{q}_n^\nu, \mathbb{R}^F \underset{\sim}{q}_m^\nu)_{\mathcal{B}}^* = (\mathbb{R}^{F'(\nu)})_{nm}^*, \tag{1.19}$$

and $(\mathbb{I}^{F'(\nu)})_{mn} = (\underset{\sim}{q}_m^\nu, \underset{\sim}{q}_n^\nu)_{\mathcal{B}} = \delta_{mn}$. The representation (1.15)-(1.16) is not far off the mark The nonzero elements of $\mathbb{L}^{F(\nu)}$ are indeed in the position of the nonzero elements of $\mathbb{R}^{F(\nu)\dagger}$ (the dagger meaning transposition and conjugation), only the normalization is not quite right. If we were to set

$$q_n^\nu(z) = a_n^\nu\, p_n^\nu(z) = a_n^\nu\, z^{n+\nu}, \qquad a_n^\nu \in \mathbb{C},\ n \in \mathbb{Z}, \tag{1.20a}$$

then (1.15) and (1.19) lead to $|a_n^\nu|^2 (n+\nu) = |a_{n-1}^\nu|^2$. As absolute values are positive real numbers, this implies first that ν must be real. As for n, two recurrence relations for $|a_n^\nu|^2$ in terms of $|a_0^\nu|^2$ may be set up, depending on whether $n > 0$ or $n < 0$:

$$|a_n^\nu|^2 = |a_0^\nu|^2\, \Gamma(\nu+1) / \Gamma(n+\nu+1), \tag{1.20b}$$

$$|a_{-n}^\nu|^2 = |a_0^\nu|^2\, \Gamma(\nu+1) / \Gamma(\nu-n+1), \quad n \geqslant 0. \tag{1.20c}$$

Equation (1.20b) yields a recursion relation, while (1.20c) is a limitant for $\nu \neq 0$ as $\Gamma(\nu-n+1)$ alternates in sign for $n > 1$, yet absolute values must be positive. It follows that only $\nu=0$ in (1.20b) will produce submatrices in (1.17) following the adjuntion property (1.19), necessary for a self-adjoint representation of w. There, an orthonormal basis is $a_n\, p_n^0(z)$, with

$$a_n = (n!)^{-1/2}\, a_0, \quad n \geqslant 0, \quad a_{-n} = 0,\ n > 0. \tag{1.20d}$$

1.11 A measure for a Hilbert space.

What is the Hilbert space? In that space, an orthonormal basis should be provided by $q_n(z) = a_n p_n^0(z) = (n!)^{-1/2} a_0 z^n$, $n \geq 0$ (we have set the phase of a_n independent of n). What is a possible inner product which would satisfy this requirement? We may follow Galbraith and Louck (15) in proposing, for $f(z)$ and $g(z)$ analytic in a common open circle containing the origin,

$$(\tilde{f},\tilde{g})_{GL} = |a_0|^{-2}\, f\,(d/dz)^*\, g(z)\,|_{z=0}. \tag{1.21a}$$

In this way, one ensures that

$$(\tilde{q}_n,\tilde{q}_{n'})_{GL} = (n!\, n'!)^{-1/2} \frac{d^n}{dz^n}\, z^{n'} \Big|_{z=0} = \delta_{n,n'}. \tag{1.21b}$$

The inner product (1.21) is very handy for computations involving quantum creation and annihilation operators, since (1.21a) is in shell-model language just $< 0|f(a)g(a^\dagger)|0 >$ while f and g are usually polynomials. Mathematicians -and many physicists- prefer inner products defined through integrals since Hilbert space theory is usually cast in that way. If we momentarily assume that g in (1.21a) satisfies also the conditions of the Fourier integral theorem -although the $q_n(z)$ clearly do not- we may write

$$\begin{aligned}(f,g)_{GL} &= |a_0|^{-2} f(d/dz)^* \frac{1}{2\pi}\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dz'\, e^{ip(z-z')} g(z')\,|_{z=0} \\ &= |a_0|^{-2} \frac{1}{2\pi}\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dz'\, f^*(ip)\, g(z') e^{-ipz'} \\ &= |a_0|^{-2} \frac{i}{2\pi}\int_{-\infty}^{\infty} dz \int_{-i\infty}^{i\infty} dz^*\, f^*(z^*) g(z) e^{-z^* z} \\ &= |a_0|^{-2} \frac{1}{\pi}\int_C d^2z\, f(z)^* g(z)\, e^{-|z|^2} = (\tilde{f},\tilde{g})_B.\end{aligned} \tag{1.22}$$

After taking $z=0$ we have changed variables to the independent $z=z'$, $z^* = ip$, then to variables $x = \mathrm{Re}\, z = \frac{1}{2}(z + z^*)$ and $y = \mathrm{Im}\, z = -i\,\frac{1}{2}(z - z^*)$ which are readily interpreted as the real and imaginary parts of a complex variable z, of which both f and g are analytic functions, the integration ranges of z and z^* are convertible into an integration over the complex $z = x + iy$ plane C, with measure $d^2z = dxdy = d\,\mathrm{Re}\, z\; d\,\mathrm{Im}\, z = |z| d|z|\; d\arg z$.

If we set $a_0 = 1$, we have exactly *Bargmann's* inner product $(\cdot,\cdot)_B$ (10, Eq. (1.6)), where (1.21b) can be easily verified, in spite of the fact that $q_n(z)$ does not abide the Fourier transform conditions. We may abandon the definition of $(\tilde{f},\tilde{g})_{GL}$ altogether and regard now only $(f,g)_B$ which poses its existence requirements quite openly: $f(z)$ must be *entire* analytic functions (i. e. no poles or branch cuts are allowed on the finite z -plane) and the growth at infinity should be overcome by the weight function $e^{-|z|^2}$. In fact, the functions may grow in some direction in C -as analytic functions do-, but may not grow faster than $\exp(\frac{1}{2}\phi\, z^2)$ for some phase ϕ as otherwise they would overwhelm the weight functions in some direction of the complex plane.

The space of entire analytic functions of growth (2,1/2) constitute -as proved by *Bargmann* (10, Sect. 1)- a separable Hilbert space under $(\cdot,\cdot)_B$. In Bargmann's space, the Fock operators (1.13) constitute thus a self-adjoint irreducible representation of the Heisenberg-Weyl algebra (1.13).

1.12 The enveloping algebra.

The construction we have just completed is straightforward, except, we should say, in our original choice of basis functions (1.14). Had we chosen a set of functions other than power functions, for example trigonometric ones, the construction of the representations of the algebra and the argument about their irreducibility and self-adjointness would have been considerably more involved. We chose power functions for the reason that we knew that $\{p_n^\nu(z)\}$ would map among themselves up and down the ladder under multiplication and differentiation: They are eigenfunctions of the operator $\mathbb{R}^F \mathbb{L}^F = z\, d/dz$ with eigenvalue $n + \nu$. The product $R\,L$ between algebra elements is undefined within the framework of the algebra. It may be incorporated through the simple device of defining a (noncommutative) product operation for the algebra elements and placing this product in a new set called the *enveloping algebra* $\bar{a}$ of the original algebra a.

If we want $\bar{a}$ to be an algebra itself, the product in $\bar{a}$ must be bidistributive with respect to the sum,

$$(c_1 X_i + c_2 X_j)\, X_k = c_1 X_i X_k = c_1 X_i X_k + c_2 X_j X_k, \tag{1.23a}$$

$$X_i (c_3 X_j + c_4 X_k) = c_3 X_i X_j + c_4 X_i X_k,\quad c_\ell \in C, \tag{1.23b}$$

and with respect to the Lie bracket it must satisfy the '*Leibnitz rule*'

$$[X_i, X_j X_k] = [X_i, X_j] X_k + X_j [X_i, X_k], \tag{1.23c}$$

which is an identity if the Lie bracket is the commutador. The enveloping algebra $\bar{a}$ of a has the structure of a ring (i.e. a non-commutative bilinear product is defined, but inverses under the product are not, and there is generally no identity under this product -although it may be defined for algebras with a centre when we take equivalence classes modulo the centre).

1.13 The Schrödinger realization.

The enveloping algebra is a very useful concept when it comes to find ways of building irreducible representations and classifying them, as the Casimir operators of semisimple algebras lie there. The properties of $N = RL = \frac{1}{2}(P^2 + Q^2 - I)$ under commutation with the elements of w are $[N,R] = R$, $[N,L] = -L$ and $[N,I] = 0$ so that if in some representation we manage to construct an eigenvector Ψ of $\mathbb{N}$ with eigenvalue μ, then by a well-known argument, $\mathbb{R}^n \Psi$ will also be an eigenvector of $\mathbb{N}$ with eigenvalue $\mu + n$, and so will $\mathbb{L}^n \Psi$ with eigenvalue $\mu - n$, unless such a vector is zero.

We can put these concepts to work on the Schrödinger realization of the Heisenberg-Weyl algebra:

$$Q \to \mathbb{Q}^S = q, \quad P \to \mathbb{P}^S = -i\frac{d}{dq}, \quad I \to \mathbb{I}^S = 1, \tag{1.24a}$$

whereby

$$R \to \mathbb{R}^S = \frac{1}{\sqrt{2}}\left(q - \frac{d}{dq}\right), \quad L \to \mathbb{L}^S = \frac{1}{\sqrt{2}}\left(q + \frac{d}{dq}\right), \quad N \to \mathbb{N}^S = \frac{1}{2}\left(-\frac{d^2}{dq^2} + q^2 - 1\right). \tag{1.24b}$$

From the arguments of the previous section, we look for the solutions $\psi^\nu(q)$ of

$$\mathbb{N}^S \psi_n^\nu(q) = (n+\nu)\psi_n^\nu(q), \quad n \in Z, \quad \mathrm{Re}\,\nu \in [0,1). \tag{1.25a}$$

These are any linear combination of

$$\begin{aligned} \Psi_n^\nu(q) &= c_n^\nu \, U(-[n+\nu] - \tfrac{1}{2}, \sqrt{2}\, q), \\ \Upsilon_n^\nu(q) &= d_n^\nu \, V(-[n+\nu] - \tfrac{1}{2}, \sqrt{2}\, q), \end{aligned} \tag{1.25b}$$

where c_n^ν and d_n^ν are arbitrary constants, and U and V are the Parabolic Cylinder functions classified in attention to their asymptotic behaviour (see 16, Sect. 19.3). They are related to the more familiar Whittaker D and U funtions through

$$U(\sigma,y) = D_{-\sigma-1/2}(y) = 2^{-\sigma/2} y^{-1/2} W_{-\sigma/2,-1/4}(y^2/2) =$$

$$= 2^{-1/4 - \sigma/2} e^{-y^2/4} U(\frac{\sigma}{2} + \frac{1}{4}, \frac{1}{2}, \frac{x^2}{2}), \tag{1.26a}$$

$$V(\sigma,y) = \pi^{-1}\Gamma(\sigma + \frac{1}{2})\{\sin \pi\sigma\, D_{-\sigma-1/2}(y) + D_{-\sigma-1/2}(-y)\}. \tag{1.26b}$$

It is an easy matter to verify that $\mathbb{R}^S$ and $\mathbb{L}^S$ indeed raise and lower the values of n in $\{\Psi^\nu_n, T^\nu_n\}$ by units (16, Sect. 19.6):

$$\mathbb{R}^S \Psi^\nu_n(q) = (c^\nu_n / c^\nu_{n+1}) \Psi^\nu_{n+1}(q), \tag{1.27a}$$

$$\mathbb{L}^S \Psi^\nu_n(q) = (n+\nu)(c^\nu_n / c^\nu_{n-1}) \Psi^\nu_n(q), \tag{1.27b}$$

$$\mathbb{R}^S T^\nu_n(q) = (n+\nu+1)(d^\nu_n / d^\nu_{n+1}) T^\nu_{n+1}(q), \tag{1.28a}$$

$$\mathbb{L}^S T^\nu_n(q) = (d^\nu_n / d^\nu_{n-1}) T^\nu_{n-1}(q). \tag{1.28b}$$

With (1.27) we are at the same point of the program for the Schrödinger realization as we were for the Fock realization, when we wrote (1.15a) for $c^\nu_n = 1$. The irreducible representations of w we obtain is thus identical to that of Eqs. (1.16) for $R^{F(\nu)}$ and $\mathbb{L}^{F(\nu)}$, but now for $R^{S(\nu)}_{(U)}$ and $L^{S(\nu)}_{(U)}$. We have now two sets of basis vectors, $\{\Psi^\nu_n(q)\}_{n\in Z}$ and $\{T^\nu_n(q)\}_{n\in Z}$, instead of the apparently single one $\{p^\nu_n(z)\}_{n\in Z}$ in (1.14). As far as the representation of the algebra is concerned, we relate them as $\mathbb{R}^{S(\nu)}_{(U)} = L^{F(\nu)\dagger}_{(U)}$ and $\mathbb{L}^{S(\nu)}_{(V)} = R^{F(\nu)\dagger}_{(V)}$. This provides the 'two' representations for the Fock case through $p^\nu_n \leftrightarrow p^\nu_{-n-1}$.

1.14 The special $\nu = 0$ cases.

Again, for $\nu=0$ we obtain from $\{\underset{\sim}{\Psi}^0_n\}_{n\in Z}$ the upper - triangular, and from $\{\underset{\sim}{T}^0_n\}_{n\in Z}$ the lower-triangular reducible indecomposable representations (1.17). The irreducible parts are provided by $\{\underset{\sim}{\Psi}^0_n\}_{n\geq 0}$ and $\{\underset{\sim}{T}^0_n\}_{n<0}$. Specifically, the $\underset{\sim}{T}^0_n$ -functions are the following:

$$\Psi^0_n(q) = c^0_n D_n(\sqrt{2}\, q) = c^0_n 2^{-n/2} e^{-q^2/2} H_n(q), \quad n = 0,1,2,... \tag{1.29a}$$

$$\Psi^0_{-n}(q) = c^0_{-n} \; 2^{n/2-1} \sqrt{\pi}\, e^{q^2/2} \; i^{n-1}\mathrm{erfc}\; q, \quad n=1,2,\ldots, \tag{1.29b}$$

where $H_n(q)$ are the Hermite polynomials and $i^m\mathrm{erfc}\; q$ the repeated integrals $\int_q^\infty dq'\; i^{m-1}\mathrm{erfc}\; q'$ of the complementary error function $i^0\mathrm{erfc}\; q = \mathrm{erfc}\; q = 2\pi^{-1/2}\int_q^\infty dq' e^{-q'^2}$. The T^0_n-functions, on the other hand, are related for $n \geqslant 0$ to the repeated integrals of Dawson's integral $F(q) = e^{-q^2}\int_0^q dq'\; e^{q'^2}$ as

$$\mathrm{T}^0_n(q) = d^0_n \; e^{-q^2/2} \{\sqrt{2}\; (d^0_{n-1})^{-1} \int_0^q dq' e^{q'^2/2} \mathrm{T}^0_{n-1}(q') - \frac{\sin(n\pi/2)}{2^{n/2}\Gamma(n/2+1)}\},$$

$$n=1,2,\ldots, \tag{1.30a}$$

$$\mathrm{T}^0_0\,(q) = d^0_0 \; 2\pi^{-1/2} \; e^{q^2/2} \; F(q), \tag{1.30b}$$

$$\mathrm{T}^0_{-n}(q) = d^0_{-n} \; 2^{1-n/2} \; \pi^{-1/2} \; e^{q^2/2} (-i)^{n-1} \; H_{n-1}(iq), \; n=1,2,\ldots \tag{1.30c}$$

For negative n' s, the $\mathrm{T}^0_{-n}(q)$ are thus simply proportional to $\Psi_n(iq)$.

We have gone into some detail in spelling out the eigenfunctions of the Schrödinger operator $\mathbb{N}^S = \mathbb{R}^S\,\mathbb{L}^S$. The reader will have recognized that (1.29a) are proportional to the well-known quantum harmonic oscilator wavefunctions, and $\mathbb{N}^S$ the corresponding *number* operator related to the Hamiltonian as $\mathbb{H}^{ho} = \mathbb{N}^S + \frac{1}{2}\,\mathbb{I}$.

1.15 Self-adjoint extensions of the number operator.

When looking for self-adjoint representations of w, one can follow the same argument which lead us to (1.18) and conclude that we must stay with the subset of Ψ^0_n and T^0_{-n-1}, $n \geqslant 0$. In that direction, the search for an appropiate Hilbert space will lead us to L^2 -spaces, where the Ψ^0_n , $n \geqslant 0$ are orthogonal and dense.

It is at least as instructive, however, to adopt a different approach which will yield results useful in other contexts as well. Since it was the number operator $\mathbb{N}^S = \mathbb{H}^{ho} - \frac{1}{2}\mathbb{I}$ which provided the basis functions (1.25) for the representations of w, let us examine the L^2-spaces where $\mathbb{N}^S$ may be self-adjoint. This is a consequence of but not a requisite for $\mathbb{Q}^S$, $\mathbb{P}^S$ and $\mathbb{I}^S$ to have this property. We recall some facts: In the $L^2(a,b)$ Hilbert space defined through an inner product $(\underset{\sim}{f},\underset{\sim}{g})_{(a,b)} = \int_a^b dq \; f(q)^* \; g(q)$, we may show that $(\underset{\sim}{f}, \mathbb{N}^S \underset{\sim}{g})_{(a,b)} = - W(\underset{\sim}{f},\underset{\sim}{g}) \Big|_a^b + (\mathbb{N}^S \underset{\sim}{f}, \underset{\sim}{g})_{(a,b)}$ holds, provided the norms of $\underset{\sim}{f}$, $\underset{\sim}{g}$, $\mathbb{N}^S \underset{\sim}{f}$ and $\mathbb{N}^S \underset{\sim}{g}$ are finite. The Wronskian $W(\underset{\sim}{f},\underset{\sim}{g}) = \underset{\sim}{f}\underset{\sim}{g}' - \underset{\sim}{g}\underset{\sim}{f}'$ valued at a and b, will be zero and $\mathbb{N}^S$ Hermitean, for spaces of functions with fixed logarithmic derivatives at those points, i.e. $h'(a) = p_a \, h(a)$ and $h'(b) = p_b \, h(b)$. Each pair of values p_a and p_b thus determines, together with certain further technical requirements related with the domain of $\mathbb{N}^S$ and its adjoint, a *self-adjoint* extension of $\mathbb{N}^S$ is determined. If $-a$ and/or b become infinity, the condition of asymptotic decrease -so that the norm of $\underset{\sim}{h}$ remain finite- takes precedence. We shall examine first the possibilities for the cas $(a,b) = (-\infty,\infty)$, then $(0,\infty)$ and last briefly, (a,b) finite.

1.16 $L^2(R)$.

The Ψ- and T-functions in (1.25b) were chosen for their asymptotic properties (16, Sect. 19.8):

$$\Psi_n^{\nu}(q) \underset{q\to\infty}{\sim} (c_n^{\nu} \; 2^{(n+\nu)/2}) \; q^{n+\nu} \; e^{-q^2/2} \quad , \qquad (1.31a)$$

$$T_n^{\nu}(q) \underset{q\to\infty}{\sim} (d_n^{\nu} \; 2^{-(n+\nu)/2} \; \pi^{-1/2}) \; q^{-n-\nu-1} e^{q^2/2} \quad . \qquad (1.31b)$$

The growing Gaussian behavior of T_n^{ν} -for all n and ν is sufficient reason to discard these as elements of the $L^2(R)$ space where $\mathbb{N}^S$ is to be self-adjoint. The next condition, at $q\to -\infty$, may be obtained through the special function relation

$$\Psi_n^{\nu}(-q) = \cos[\pi(n+\nu)] \, \Psi_n^{\nu}(q) + [\pi c_n^{\nu}/d_n^{\nu} \; \Gamma(-n-\nu)] \, T_n^{\nu}(q), \qquad (1.31c)$$

and a similar one for $T_n^{\nu}(-q)$ which we omit as it is now unnecessary. From (1.31a) and (1.31c) it follows that for $q\to\infty$, only when the second summand in (1.31c) is zero will $\Psi_n^{\nu}(-q)$ be in $L^2(R)$. The finitude of $\Psi_n^{\nu}(q)$ is proven as an immediate consequence of (1.25a), since $\mathbb{N}^S$ has no singularities for finite q. This means that $-n-\nu = 0, -1, -2, \ldots$ i. e. that $\nu=0$ and n is a non-negative integer. The end result: Only $\{\Psi_n^0(q)\}_{n\geq 0}$ as given in (1.29a) may serve as basis functions for a self-adjoint representation of w on $L^2(R)$. This set is closed under the algebra (c. f. Eqs. (1.27a), (1.27b)) and functional analysis tells us that the closure of the linear hull of $\{\underset{\sim}{\Psi}_n^0\}_{n\geq 0}$ is $L^2(R)$. This *quantizes* the problem through restricting the spectrum of $\mathbb{N}^S$ to be the set of nonnegative integers. We thus reconstitute the representation afforded by the Fock realization in Bargmann's Hilbert space. The unitary equivalence of the two will be implemented in Sect. 1.18.

1.17 $L^2(R^+)$.

Let us now turn to $L^2(0,\infty)$ spaces where the dense subset of once-differentiable functions have fixed logarithmic derivative at the origin, i. e. where $h'(0) = p\, h(0)$. The integrability conditions demanded by asymptotic behaviour still demand that only the $\Psi_n^\nu(q)$ appear. We find from special-function tables that

$$\Psi_n^\nu(0) = c_n^\nu\, 2^{(n+\nu)/2}\, \pi^{1/2} / \Gamma(\tfrac{1}{2}[1-n-\nu]), \tag{1.32a}$$

$$d\Psi_n^{\nu}(q)/dq|_{q=0} = - c_n^\nu\, 2^{(n+\nu)/2+1} \pi^{1/2}\, \Gamma(\tfrac{1}{2}[-n-\nu]). \tag{1.32b}$$

The basis of the $p=0$ space of functions whose derivative vanishes at the origin are $\Psi_n^{\nu}(q)$ such that (1.32b) is zero. The Gamma function provides this behaviour through its poles at $\frac{1}{2}[-n-\nu] = 0, -1, -2, \ldots$ This means $\nu=0$ and $n=2N$ where $N=0,1,2,\cdots$. The even-n harmonic oscillator wavefunctions in (1.29a) are thus obtained. Next, the basis of the $p=\infty$ space of functions which vanish at the origin require that (1.32a) be zero. Again, this is provided by the Gamma function for $\frac{1}{2}[1-n-\nu] = 0, -1, -2, \ldots$; i. e. $\nu=0$ and $n=2N+1$ where $N=0,1,2,\cdots$. These are the odd-n harmonic oscillator wavefunctions. Lastly, for fixed, finite p, $\Psi_n^{\nu\,\prime}(0) = p\, \Psi_n^{\nu}(0)$ implies the equality

$$\Gamma(\tfrac{1}{2}[1-n-\nu]) = -(2p)^{-1}\, \Gamma(\tfrac{1}{2}[-n-\nu]). \tag{1.33}$$

This is a transcendental equation whose set of solutions for $n+\nu$ gives the spectrum of the self-adjoint extension of $\mathbb{N}^S$ determined by p. It is not difficult to see that for p positive, the solutions $n+\nu$ are all positive and that only one solution exists between any two consecutive integers. Similarly, we check that if some $n+\nu$ is a solution of (1.33), no $(n+M)+\nu$ with M integer may be a solution to to the same equation. For p negative, the spectrum has both positive and negative values, but the spacing property is the same.

Consequence: *Only* the $p=0$ and $p=\infty$ self-adjoint extensions of $\mathbb{N}^S$ have spectra with equally-spaced eigenvalues. Enter (1.27). The elements of w take us between functions whose $n+\nu$ differs by units or, as $\mathrm{Re}\,\nu \in [0,1)$, whose n differs by units. When applied in spaces determined by self-adjoint extensions of $\mathbb{N}^S$, the elements of w will *not* respect the constant p which therefore *cannot* be used to classify irreducible representations. Repeated application of $\mathbb{R}^S$ on the $p=0$ $\Psi_0^0(q)$ function will take us, at each step, between $p=0$ and $p=\infty$ functions. A self-adjoint representation of w requires thus the union of these two self-adjoint extension spaces of $\mathbb{N}^S$. In fact, when the basis functions are extended to all of R as even and odd functions, we obtain the previous $L^2(R)$ space. There, in addition, $\mathbb{Q}^S$ and $\mathbb{P}^S$ are not only hermitean, but self-adjoint. The hermiticity properties of $\mathbb{N}^S$ in the union of the $p=0$ and $p=\infty$ spaces is a consequence of the presence of functions which vanish at the origin, so that the crossed-p Wronskian continues to be zero. For any other self-adjoint extension space (determined by fixed, finite p) this is not so: The repeated application of the elements of w to some function in that space will never take us back to that same space.

We thus conclude that in order that an algebra of operators have a self-adjoint representation, it is necessary but not sufficient that the operators chosen to classify the basis functions be self-adjoint. For semisimple groups these are usually the Casimir operators.

The case worked out above makes it unnecessary to further analyze $L^2(a,b)$ for a and b finite: The spectrum of $\mathbb{N}^S$ is never equally spaced, as moreover, it asymptotically resembles the n^2 -spectrum of an impenetrable box. This argument will also eliminate from consideration any other self-adjoint extensions for the elements of the Lorentz algebra built out of $\mathbb{N}^S$, $(\mathbb{R}^S)^2$ and $(\mathbb{L}^S)^2$, which are elements in the universal covering algebra $\bar{w}$ of w.

1.18 The Bargmann transform.

The last point in this chapter will be to relate the Fock realization in Bargmann Hilbert space, Eqs. (1.13) and (1.22), with the Schrödinger realization in the $L^2(R)$ Hilbert space. Both of these are separable and hence should be unitarily equivalent. The integral kernel $A(z,q)$ which relates them as

$$f^B(z) = \int_R dq\, A(z,q) f(q) \in B, \qquad (1.34a)$$

$$f(q) = \int_C e^{-|z|^2} d^2z\, A(z,q)^* f^B(z) \in L^2(R), \qquad (1.34b)$$

may be found (10, Introduction) through requiring that if $f(q) \in C^2 \cap L^2(R)$ is mapped on $f^B(z)$, then $\mathbb{R}^S f(q)$ and $\mathbb{L}^S f(q)$ be mapped on $\mathbb{R}^F f^B(z)$ and $\mathbb{L}^F f^B(z)$ respectively. This leads to a set of two coupled first-order differential equations whose solution was given by Bargmann (10). An equivalent solution may be found as a generating function built out of two dense orthonormal bases: Bargmann-normalized power functions $\{q_n^0\}_{n\geqslant 0}$ in (1.20a)-(1.20d) with $a_0 = 1$ for B, and the harmonic oscillator wavefunctions $\{\Psi_n^0\}_{n\geqslant 0}$ in (1.29a) normalized with $c_n^0 = (\pi^{1/2} n!)^{-1/2}$ for $L^2(R)$:

$$A(z,q) = \sum_{n=0}^{\infty} q_n^0(z)\, \Psi_n^0(q)^*$$

$$= e^{-q^2/2}\, \pi^{-1/4} \sum_{n=0}^{\infty} (n!\, 2^{n/2})^{-1}\, z^n H_n(q) \qquad (1.35)$$

$$= \pi^{-1/4} \exp\left[-\tfrac{1}{2}(z^2+q^2)+\sqrt{2}\, z\, q\right],$$

where we have made use of the Hermite polynomial generating function.

CHAPTER 2: A LIE GROUP.

Out of the Heisenberg-Weyl Lie algebra of last Chapter we shall develop the Heisenberg-Weyl Lie group through the exponential map. The basic concepts of harmonic analysis on the group and coset manifolds will follow. Out of these we shall find various equivalent unitary irreducible representations through infinite matrices and integral kernels.

2.1 Ado's theorem and the exponential map.

A theorem by Ado (17) states that every Lie algebra a over C is isomorphic to some matrix algebra. That is, if we have a finite dimensional algebra we can find a finite dimensional faithful $N \times N$ matrix representation which will be a subalgebra of $gl(N,C)$. For the Heisenberg-Weyl algebra w defined in the first chapter through (1.1), a faithful representation and subalgebra of $gl(3,C)$ is given by (1.9); w is *not* contained in $gl(2,C)$.

We can use this representation in order to define the *exponential map* of a into a Lie group G, which will be a subgroup of $GL(N,C)$, the group of $N \times N$ nonsingular, complex matrices. If a has a vector basis $\{X_k\}_{k=1}^{D}$ faithfully represented by $N \times N$ matrices $\{\mathsf{X}_k^F\}_{k=1}^{D}$, then to every element $X=\Sigma_{k=1}^{D} x_k X_k$, $x_k \in C$ represented by $\mathsf{X}= \Sigma_{k=1}^{D} x_k \mathsf{X}_k$, we associate the matrix $\mathsf{G}(x_1,\ldots,x_D)= exp\, \mathsf{X}$, element of $GL(N,C)$ and faithful analytic representation of the Lie group G. The parameters $\{x_k\}_{k=1}^{D}$ constitute the *canonical coordinate* system of G. The exponential map exists since - it is easy to show- the exponential of an arbitrary $N \times N$ matrix with finite elements is an absolutely convergent series which yields another such matrix which is, moreover, invertible, (as $det\, exp\, \mathsf{X}=exp\, tr\, \mathsf{X}$) and sends the zero element in a to the unit element in G. Lastly, the matrix elements of $exp\, \mathsf{X}$ are analytic functions of the canonical coordinates. They thus satisfy one last requirement of representations of topological groups, namely that these be continuous maps of the abstract group into a matrix subgroup of $GL(N,C)$.

2.2. The Heisenberg-Weyl group.

The algebra w with general element represented by X^δ in (1.9) is easy to exponentiate since the matrices are nilpotent. We have the elements of W represented by 3×3 matrices as

$$\mathsf{G}(x,y,z) = exp\, i(x\mathsf{Q}^\delta + y\mathsf{P}^\delta + z\mathsf{I}^\delta) = exp\, i\begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ y & iz & 0 \end{pmatrix} \qquad (2.1)$$

$$= I + i\begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ y & iz & 0 \end{pmatrix} - \frac{1}{2!}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & xy & 0 \end{pmatrix} = \begin{pmatrix} 1 & ix & 0 \\ 0 & 1 & 0 \\ iy & -z-\frac{xy}{2} & 1 \end{pmatrix}.$$

$$= \exp\,(ix\,Q^{\delta})\,\exp\,(iy\,P^{\delta})\,\exp(\,i\,[z + xy/2]\,I^{\delta}) \qquad (2.1\ \text{cont.})$$

$$= \exp\,(iy\,P^{\delta})\,\exp\,(ix\,Q^{\delta})\,\exp(\,i\,[z - xy/2]\,I^{\delta}).$$

The last two lines go under the name of the *Weyl commutation relations* (18). Through introduction of the i in the exponential map we are assuring that self-adjoint representations of w exponentiate to unitary representations of W.

The matrix representation (2.1) yields the composition law of the abstract group elements $g(x,y,z) \in W$ as

$$g(x_1,y_1,z_1)g(x_2,y_2,z_2) = g(x_1+x_2, y_1+y_2, z_1+z_2+\tfrac{1}{2}[y_1x_2-x_1y_2]), \qquad (2.2a)$$

$$e = g(0,0,0)\ ,\ g(x,y,z)^{-1} = g(-x,-y,-z). \qquad (2.2b)$$

Associativity clearly holds. Finally, w is the Lie algebra of W, since

$$\partial G(x,y,z)/\partial x\,\Big|_{g=e} = i\,Q,\ \partial G(x,y,z)/\partial y\,\Big|_{g=e} = i\,P,\ \partial G(x,y,z)/\partial z\,\Big|_{g=e} = i\,I. \qquad (2.3)$$

All parameters range over R, the group manifold is thus isomorphic to R^3, non-compact (Sect. 2.19) and simply connected. The centre of W generated by I is the subgroup of elements $g(0,0,z)$.

2.3 Functions on the group and on coset spaces.

We consider now complex-valued functions $f(g) = f(x,y,z)$ on R^3 identified as the W manifold. We may act on this space of functions with $g' \in W$ through an action from the *right*

$$f(g) \xrightarrow{g'(R)} f^{R}_{g'}(g) = f(g\,g'), \qquad (2.4a)$$

and an action from the *left*

$$f(g) \xrightarrow{g'(L)} f^{L}_{g'}(g) = f(g'^{-1}g). \qquad (2.4b)$$

In this rather trivial way, W becomes a *Lie transformation group* on the differentiable manifold $M = W$.

The general definition of a Lie transformation group is such that to each pair (p,g'), $p \in M$ (a differentiable manifold), $g' \in G$ (a Lie group), there is associated an element $p_{g'} \in M$ (which may be denoted and mean pg' or g'^{-1} according to convenience), such that

a) the map is differentiable
b) $p\,e = p$ or $e\,p = p$
c) $(p\,g_1)g_2 = p(g_1\,g_2)$ or $g_2^{-1}(g_1^{-1}\,p) = (g_1 g_2)^{-1}\,p$.

These axioms are the result of the group axioms when M is G itself, as above. But they also hold when M is a *coset space*: if $H \subset G$, we consider the points of M to be the sets $p^L = Hg$ or $p^R = gH$. There is a standard argument to show that any two of these sets are either disjoint or they coincide, and that they *partition* G. They are called $M^L = H\backslash G$ (left cosets) or $M^R = G/H$ (right cosets) respectively.

Left cosets map into each other under the *right* action of the group, i. e. $p^L = Hg \xrightarrow{g'(R)} (Hg)g' = H(gg') = p^{L'}$. *Right* cosets map under the *left* action of the group $p^R = gH \xrightarrow{g'(L)} g'^{-1}(gH) = (g'^{-1}g)H = p^{R'}$.

Consider our example $G=W$ and $H_Q = \{g(x,0,0)\}_{x\in R}$. Since

$$g(x,y,z - \tfrac{1}{2}\,xy) = g(x,0,0)\;g(0,y,z), \tag{2.5a}$$

we may partition W into cosets by H_Q letting $g(x,0,0)$ range over H_Q (i. e. x over R):

$$c_Q^L\,(y,z) = \{g(x,0,0)\}_{x\in R}\;g(0,y,z) = \{g(x,y,z - \tfrac{1}{2}\,xy)\}_{x\in R}. \tag{2.5b}$$

The manifold $M = H_Q\backslash W$ is then isomorphic to R^2 and its representative elements may be labelled by $g(0,y,z)$. The space of left cosets $M = H_Q\backslash W$ may be subject to transformations through the right action of W as

$$\begin{aligned} c_Q^L(y,z) &\xrightarrow{g(x',y',z')\ (R)} c_Q^L(y,z)g(x',y',z') \\ &= \{g(x,0,0)\}_{x\in R}\;g(0,y,z)\;g(x',y',z') \\ &= \{g(x,0,0)\}_{x\in R}\;g(x',y+y',\;z+z'+\tfrac{1}{2}\,x'y) \qquad (2.5c) \\ &= \{g(x,0,0)\}_{x\in R}\;g(x',0,0)g(0,y+y',z+z'+\tfrac{1}{2}x'y'+x'y) \\ &= c_Q^L\,(y+y',\;z+z'+\tfrac{1}{2}\,x'y'+x'y). \end{aligned}$$

2.4 Transitive and effective action.

The action of W on the right on $M = H_Q\backslash W$ given by (2.5c) is transitive and effective. We recall these concepts. A group G acts *transitively* on a manifold M when for any two points $p, \bar{p} \in M$ there exists a $g \in G$ which maps p on $\bar{p}$, i. e. $\bar{p} = p_g$. In our example above, given two cosets $c_Q^L(y,z)$ and $c_Q^L(\bar{y},\bar{z})$, the $g \in W$ which does the job is any $g(x', \bar{y}-y, \bar{z}-z-\frac{1}{2}x'(\bar{y}+y)), x' \in R$.

A differentiable manifold where a Lie group acts transitively is a *homogeneous* space for the group. Every coset space of a Lie group is homogeneous for it, and in fact, every homogeneous space is a coset space for these groups (19, Chapter II, Theorem 3.2).

The *isotropy group* (also called *stability* or *little* group) of any fixed point $p \in M$ is the set of elements $h \in I_p \subset G$ such that $p_h = p$. In the previous paragraph, setting $\bar{y} = y$ and $\bar{z} = z$, the isotropy group of $c_Q^L(y,z) \in H_Q\backslash W$ is $I_{c(y,z)} = \{g(x',0,x'y)\}_{x' \in R}$. The isotorpy group, quite clearly, may depend on the coordinates of p. In fact, if G is transitive over M, the isotropy groups of any two points in M are isomorphic and conjugate: $I_{pg} = g^{-1} I_p g$.

The action of G on M is said to be *effective* if no transformation in G except the identity leaves all of the points in M fixed. That is $\bigcap_{p \in M} I_p = \{e\}$ and is the case in the example(2.5c). It would *not* have been the case had we chosen $H_I = \{g(0,0,z)\}_{z \in R}$ to define the coset space $H_I\backslash W = W/H_I$ with cosets $c_I(x,y)$. There the non-abelianity of W would have been completely lost as H_I would be the isotropy group for every coset $c_I(x,y)$. It is straightforward to show that a group G acts effectively on a coset space $H\backslash G$ if and only if H does not contain a normal subgroup N of G. (A normal subgroup $N \lhd G$ we recall, is a subgroup of G such that $gn\,g^{-1} = n' \in N$ for all $n \in N$; in our example H_I is normal in W as well as its centre.) The proof goes as follows : Every $n \in N \subset H \subset G$ applied to a coset $H\,g$ yields $Hgn = Hn'g = Hg$ independently of the coset and hence is the isotropy group for the whole coset space. Conversely, if some n exist such that $Hgn = Hg$ for all g, they form a subgroup $N \subset G$; since $Hg\bar{g}$ is a coset $Hg\bar{g}n\bar{g}^{-1} = Hg$ it follows that $N \lhd G$.

2.5 Multiplier representations.

Normal or central subgroups cannot be used to divide the group into coset spaces without loosing the effectiveness of the group action on the coset manifold, but we may use them to obtain *multiplier* representations. This is a subject closely related to induced representations (20, Chapters 16 and 17; 21, Chapter 9) which we will apply to the Heinsenberg-Weyl case. The point in that, the group N being normal in G , there is a natural action of G on N given by $n \xrightarrow{g} gng^{-1} = n_g \in N$. If we have a manifold $M = H\backslash G$ where the action of G is effective, we may build $N\backslash M = N \cdot H\backslash G$ where the action of G is no longer effective, and supplement it with a more intimate knowledge of the group N , its representations in particular.

We may apply this principle to our example in considering functions over the space of cosets $c_Q^L(y,z)$ of the special form

$$F^{\lambda}\ (c^{L}_{Q}(y,z)) = f^{\lambda}(y)\ e^{i\lambda z},\quad \lambda \in C. \tag{2.6}$$

The coset space is effectively acted upon by W as shown in (2,5c), but the factorized form (2.6) and the number λ in it are respected, as the points (y,z) move under $g(x',y',z')$ to $(y+y', z+z'+\frac{1}{2}x'y'+x'y)$. This induces a transformation of the functions on y (functions on $H_I\ H_Q\backslash W$)

$$f^{\lambda}(y) \xrightarrow{g(x',y',z')} f^{\lambda}_{g'}(y) = f^{\lambda}(y+y')\exp\left[i\lambda(z'+\tfrac{1}{2}\,x'y'+x'y)\right]. \tag{2.7}$$

The factor $\mu(y,g) = e^{i\lambda\cdots}$ is called a *multiplier* factor. It is a result of function theory that there is no proper subspace of $L^2(R)$ which is invariant under the action of all translation and multiplication-by-exponential operators. The action of (2.7) is thus irreducible on $L^2(R)$, since it cannot be broken into invariant proper subspaces.

2.6 Infinitessimal generators in 3, 2 and 1 variable.

We have seen the action of W on three-variable functions on W in (2.4), on two-variable functions on $H_Q\backslash W$ in (2.5), and on one-variable functions with multiplier in (2.6)-(2.7). We shall show what this means in terms of the Lie algebra generator realization. We consider transformations $g(\delta x',\delta y',\delta z')$ near to the group indentity $g(0,0,0)$ and collect Taylor expansion terms to first order in δ, so that

$$f(p) \xrightarrow{\delta g'} f(p_{\delta g'}) = (1+i\,[\,\delta x'\mathbb{Q}^M+\delta y'\,\mathbb{P}^M+\delta z'\,\mathbb{I}^M] + o(\delta^2))\,f(p). \tag{2.8}$$

The coordinates of $p \in M$ may be three, two or one, so that $\mathbb{Q}^M$, $\mathbb{P}^M$ and $\mathbb{I}^M$ will be the generator of the group of transformations on that manifold.

Setting (2.2) into the action from the right in the 3-variable case we obtain

$$\mathbb{Q}^{W(R)} = -i(\partial_x+\tfrac{1}{2}\,y\partial_z),\ \mathbb{P}^{W(R)} = -i(\partial_y - \tfrac{1}{2}x\partial_z),\ \mathbb{I}^{W(R)} = -\,i\partial_z. \tag{2.9a}$$

A similar procedure for the action from the left (2.4b) yields

$$\mathbb{Q}^{W(L)} = i(\partial_x - \tfrac{1}{2}\,y\partial_z),\ \mathbb{P}^{W(L)} = i\,(\partial_y+\tfrac{1}{2}x\partial_z),\ \mathbb{I}^{W(L)} = i\partial_z. \tag{2.9b}$$

The generators (2.9a) commute with those in (2.9b). From (2.5c), for $H_Q\backslash W$ we find

$$\mathbb{Q}^{H_Q\backslash W} = -iy\partial_z,\ \mathbb{P}^{H_Q\backslash W} = -i\partial_y,\ \mathbb{I}^{H_Q\backslash W} = -i\partial_z. \tag{2.9c}$$

Clearly, we can repeat this procedure for $H_P\backslash W$, W/H_Q and W/H_P. Finally, for (2.7)

$$\mathbb{Q}^{(\lambda)} = \lambda y,\ \mathbb{P}^{(\lambda)} = -i\partial_y\ ,\ \mathbb{I}^{(\lambda)} = \lambda\ . \tag{2.9d}$$

The last set of generators of W constitute the Schrödinger realization of w given in (1.24a) for $\lambda y = q$. We can identify λ with $\hbar$, which is assigned a particular value by Nature.

Perhaps the most striking feature of the generators (2.9d) stemming from the multiplier action (2.7) is that Sophus Lie would not have recognized them as generators of a Lie algebra (22). He considered Lie transformation groups, where groups act effectively on manifolds, so his generators are all and only first-order differential operators (with coefficients which are functions of the coordinates). Generators with zeroth order operators, as $\mathbb{Q}^\lambda$ and $\mathbb{I}^\lambda$ above, stem from multiplier group action (23).

Lie algebras of operators of order higher that first are interesting in physics. The associated Lie groups are generally groups of integral transforms (12, 13, 14 Part 4), rather than manifold mappings of the type (2.4).

2.7 On representations of groups on homogeneous spaces.

A representation ρ of a group G on a vector space V is defined in a very similar way as those of an algebra, namely, as a homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$ from the group G into the group $\mathrm{GL}(V)$ of linear operators on V. The homomophism means that $\rho(g_1)\rho(g_2) = \rho(g_1 g_2)$ and $\rho(e) = 1$. One is generally interested in the case when V is a separable Hilbert space and when the representation is strongly continuous, i. e. $\|\rho(g)f - \rho(g_0)f\| \to 0$ as $g \to g_0$.

The physicists' experience with compact groups, where the V are finite-dimensional, leads one to search for matrix representations which may be of infinite dimension, or for their generalizations as integral kernels. A finite-dimensional representation on $V = R^3$ is already provided by (2.1), although it is non-unitary. An appropiate Hilbert space may be constructed for functions of three, two or one variable on which we defined the action of W.

Infinite-dimensional matrix representations may be obtained if we give a complete denumerable basis for the separable Hilbert space, $\{\Psi_n(y)\}_{n=0}^{\infty}$, and build the matrix $\rho(g) = \|\mathcal{D}_{nn'}(g)\|$ with elements $\mathcal{D}_{nn'}(g) = (\Psi_n, \Psi_{n'g})$, the inner product being that of $L^2(R)$ and the group action given by (2.7). Similarly, integral-kernel representations may be obtained through working with a Dirac-orthonormal basis for the space, $\{\chi_\nu(y)\}_{\nu \in s}$, where s is some index interval. There, the integral kernels are $\underset{\sim}{\rho}(g) = \|\mathcal{D}_{\nu\nu'}(g)\|$ with $\mathcal{D}_{\nu\nu'}(g) = (\chi_\nu, \chi_{\nu' g})$.

The one-variable $L^2(R)$ space the is manifestly mapped onto itself by W. Its choice is dictated over any other $L^2(a,b)$ by the fact that the group action (2.7) includes translations in the argument y, so that only spaces of periodic functions of period $b-a$ may be contemplated, but the exponential factor $e^{i\lambda x' y}$ does not respect this periodicity over any finite interval.

2.8 The Q-subgroup basis.

Let us start first with the integral-kernel representations afforded by the normalized Dirac eigenbasis of the 'position' operator $\mathbb{Q}^{(\lambda)}$,

$$\mathbb{Q}^{(\lambda)} \chi_q^\lambda(y) = q\, \chi_q^\lambda(y) \;, \quad q,\ \lambda \in R, \tag{2.10a}$$

$$\chi_q^\lambda(y) = |\lambda|^{1/2}\, \delta(q - \lambda y\,). \tag{2.10b}$$

Then, the representation of W labelled by $\lambda \in R$ in the $\mathbb{Q}$-eigenbasis is

$$\begin{aligned} D_{qq'}^{\lambda,(Q)}(g(x',y',z')) &= (\chi_q^\lambda, \chi_{q'g}^\lambda) \\ &= \int_{-\infty}^{\infty} dy\, \chi_q^\lambda(y)^* \, \chi_{q'}^\lambda(y+y')\exp[i\lambda(z'+\tfrac{1}{2}x'y'+x'y)] \\ &= \delta\,(q-q'+\lambda y')\ \exp i[\lambda(z'+\tfrac{1}{2}x'y')+x'q\,]. \end{aligned} \tag{2.11}$$

The representation property holds:

$$\int_{-\infty}^{\infty} dq'\, D_{qq'}^{\lambda(Q)}(g_1)\, D_{q'q''}^{\lambda(Q)}(g_2) = D_{q\,q''}^{\lambda(Q)}(g_1 g_2), \tag{2.12}$$

and $D_{qq'}^{\lambda(Q)}(e) = \delta(q-q')$ is the unit operator. The strong continuity of (2.11), meaning the strong continuity of (2.7) is easily ascertained. The subgroup $g(x',0,0)$ is represented by a 'diagonal' integral kernel (i. e. one with a factor of $\delta(q-q')$). This representation diagonalizes the subgroup generated by Q.

A representation of the operators in the Lie algebra w and W through integral kernels may be obtained subjecting the integral kernel (2.11) to the limiting procedure in (2.8):

$$Q_{q\,q'}^{\lambda(Q)} = -\,i\partial_{x'}\, D_{qq'}^{\lambda(Q)}\,(g(x',y',z'))|_{g=e} = q\ \delta(q-q'), \tag{2.13a}$$

$$P_{q\,q'}^{\lambda(Q)} = -\,i\partial_{y'}\, D_{qq'}^{\lambda(Q)}\,(g(x',y',y'))|_{g=e} = -i\lambda\delta'(q-q'), \tag{2.13b}$$

$$I^{\lambda(Q)}_{q\,q'} = -i\,\partial_{z'}\, D^{\lambda(Q)}_{q\,q'}\,(g(x',y',z'))|_{g=e} = \lambda\,\delta(q-q'). \qquad (2.13c)$$

These integral kernels represent the differential operators of the Schrödinger realization on the space of differentiable functions in $L^2(R)$, a space dense in the latter.

2.9 The P -subgroup basis.

A second generalized Dirac basis of $L^2(R)$ is provided by the generalized eigenfuctions of $\mathbb{P}^{(\lambda)}$ in (2.9c):

$$\mathbb{P}^{(\lambda)}\tilde{\chi}^{\lambda}_{p}\,(y) = p\,\tilde{\chi}^{\lambda}_{p}\,(y), \quad p,\ \lambda \in R, \qquad (2.14a)$$

$$\tilde{\chi}^{\lambda}_{p}\,(y) = (2\pi)^{-1/2}\,e^{ipy}. \qquad (2.14b)$$

The representation thus obtained is

$$D^{\lambda(p)}_{p\,p'}\,(g(x',y',z')) = (\tilde{\chi}^{\lambda}_{p}\,, \tilde{\chi}^{\lambda}_{p'g})$$

$$= \delta\,(p-p'-\lambda x')\ \exp i[\lambda(z'+\tfrac{1}{2}\,x'y') + p'y'] \qquad (2.15)$$

$$= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dq'\ (\tilde{\chi}^{\lambda}_{p}\,, \chi^{\lambda}_{q})\ D^{\lambda(Q)}_{qq'}\,(g)\,(\chi^{\lambda}_{q'}, \tilde{\chi}^{\lambda}_{p'}).$$

It satisfies the representation property (2.12), and diagonalizes the subgroup $g(0,y,0)$ generated by P. The last line in Eq. (2.15) shows that the $D^{\lambda(P)}_{pp'}\,(g)$ are *equivalent* to the $D^{\lambda(Q)}_{qq'}\,(g)$ in (2.11), since they are basically the double Fourier transforms of the $D^{\lambda(Q)}_{qq'}\,(g)$ in (2.11), as $(\chi^{\lambda}_{q},\ \tilde{\chi}^{\lambda}_{p}) = (2\pi|\lambda|)^{-1/2}\ e^{ipq/\lambda}$ is the Fourier transform kernel. The *overlap* coefficients between the two eigenbases (2.10) and (2.14) yield the tranformation kernel between the two representations (2.11) and (2.15). We shall return to this point below when we comment on the unitarity and completeness of the representation set. Meanwhile, let us produce further representations equivalent to the above ones.

The Lie algebra representation obtained from (2.15) may be written as

$$Q^{\lambda(P)}_{p\,p'} = i\,\lambda\delta'(p-p'), \quad P^{\lambda(P)}_{pp'} = p\delta\,(p-p'), \quad I^{\lambda(P)}_{pp'} = \lambda\delta(p-p') \qquad (2.16)$$

which, again, is only the Fourier transform of the former.

2.10 The quantum free-fall nonsubgroup basis.

Further generalized bases of $L^2(R)$, unrelated to subgroups of W may be produced as eigenbases of self-adjoint operators in the enveloping algebra of w. Quantum Mechanics has a large supply of such operators. Let us start with the free-fall Schrödinger Hamiltonian and its Dirac-orthonormal eigenfunctions:

$$\mathbb{H}^{\ell}\Lambda^{\lambda}_{\alpha}(y) = [\tfrac{1}{2}\mathbb{P}^{(\lambda)2} + \mathbb{Q}^{(\lambda)}]\,\Lambda^{\lambda}_{\alpha}(y) = \alpha\Lambda^{\lambda}_{\alpha}(y)\ ,\quad \alpha \in R \tag{2.17a}$$

$$\Lambda^{\lambda}_{\alpha}(y) = (2|\lambda|^{-1/2})^{1/3}\quad \mathrm{Ai}([2\lambda]^{1/3}(y-\alpha/\lambda)), \tag{2.17b}$$

where $\mathrm{Ai}(z)$ is the Airy function of the first kind (14, subsect. 9.5.3). Then, a calculation aided by the Fourier transform shows that

$$D^{\lambda(\ell)}_{\alpha\alpha'}(g(x',y',z')) = e^{-i\pi/4}(2\pi x')^{1/2}\,|\lambda|^{-1}\ \times$$
$$\times\ exp\ i[\lambda z' + \tfrac{1}{2}x'(\alpha+\alpha') + \tfrac{1}{2x'}(y' + \tfrac{\alpha-\alpha'}{\lambda})^2 - \tfrac{1}{24}\lambda^2 x'^3\ ]. \tag{2.18}$$

Rather tediously, we can verify that the analogue of the representation property (2.12) holds. Using

$$\underset{\varepsilon \to 0^+}{\mathrm{l.i.m.}}\ e^{-i\pi/4}(2\pi\varepsilon)^{-1/2}\ exp\ i(s^2/2a^2\varepsilon) = |a|\delta(s), \tag{2.19}$$

we can show that (2.18) is indeed $\delta(\alpha-\alpha')$ for $g=e$, as well as some further properties which will be seen later. (If the powers of λ throughout produce any distress, the reader may check that "units" of λ are properly given if the arguments of transcendental functions are to be dimensionless, while wavefunctions have units of $(lenght)^{-1/2}$ and integral kernels of $(lenght)^{-1}$.

The task of finding the Lie algebra of integral kernels now takes us to find through (2.19) and its derivatives with respect to s, as in (2.13), the integral kernel representations:

$$Q^{\lambda(\ell)}_{\alpha\alpha'} = \tfrac{1}{2}\lambda^2\delta''(\alpha-\alpha') + \alpha\delta(\alpha-\alpha'), \tag{2.20a}$$

$$P^{\lambda(\ell)}_{\alpha\alpha'} = -i\ \lambda\ \delta'\ (\alpha-\alpha'), \tag{2.20b}$$

$$I^{\lambda(\ell)}_{\alpha\alpha'} = \lambda\ \delta\ (\alpha-\alpha'). \tag{2.20c}$$

Note that, indeed,

$$\left(\frac{1}{2} P^2 + Q \right)^{\lambda(\ell)}_{\alpha\alpha'} = \alpha\, \delta(\alpha-\alpha') \qquad (2.20d)$$

is a diagonal *number* integral kernel, as it ought to be from (2.17a).

2.11 The quantum harmonic oscillator basis.

A most convenient denumerable orthonormal basis is provided by the eigenfunctions of the quantum harmonic oscillator Hamiltonian

$$\mathbb{H}^n\, \Psi^\lambda_n(y) = \frac{1}{2}\left[\mathbb{P}^{(\lambda)^2} + \mathbb{Q}^{(\lambda)^2} \right] \Psi^\lambda_n(y) = |\lambda|\left(n + \frac{1}{2}\right) \Psi^\lambda_n(y), \qquad (2.21a)$$

$$\Psi^\lambda_n(y) = \left(2^n n! [\pi/|\lambda|]^{1/2}\right)^{-1/2} e^{-|\lambda| y^2/2}\, H_n(|\lambda|^{1/2} y) \qquad (2.21b)$$

$$n = 0,\ 1,\ 2, \ldots \ .$$

In this basis we must calculate

$$\mathcal{D}^{\lambda(h)}_{nn'}(g(x',y',z')) = \int_{-\infty}^{\infty} dy\, \Psi^\lambda_n(y)^* \Psi^\lambda_{n'}(y+y') \exp[i\lambda(z' + \tfrac{1}{2}x'y' + yx')],$$

$$(2.22a)$$

which may be done through multiplying (2.19a) by $2^{(n+n')/2}(n!n!)^{-1/2} \times$ $\times\, s^n t^{n'}$, summing over n and n' so as to use the known generating functions for the Hermite polynomials (16, Eq. 22.9.17), integrating, and finally using the generating function (3, Eq. (2.42))

$$\exp(ab+ac-bd) = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} L^{(n-n')}_{n'}(cd)\, a^n b^{n'} c^{n-n'}/n! \ ,$$

in order to separate powers of s and t. Setting $a=\sqrt{2s}$, $b=\sqrt{2t}$ and $c=(|\lambda|/2)^{1/2}(-y'+ix') = d^*$ we obtain the result

$$\mathcal{D}^{\lambda(h)}_{nn'}(g(x',y',z')) = \exp \lambda(iz + \tfrac{1}{4}[x'^2 + y'^2])\, (n'!/n!)^{1/2} \times$$

$$\times \left([|\lambda|/2]^{1/2} [-y'+ix']\right)^{n-n'} L^{(n-n')}_{n'}\left(\tfrac{1}{2}|\lambda|[x'^2 + y'^2]\right),$$

$$(2.22b)$$

valid for $n \geqslant n'$. Similarly, for $a = \sqrt{2t}$, $b = \sqrt{2s}$ and $c = (|\lambda|/2)^{1/2}(y + ix) = d^*$ we obtain

$$\mathcal{D}^{\lambda(h)}_{nn'} \quad (g(x',y',z')) = \exp \lambda(iz + \frac{1}{4}[x'^2 + y'^2])(n!/n'!)^{1/2} \quad \times$$

$$\times \quad ([|\lambda|/2]^{1/2}[y' + ix'])^{n'-n} \; L_n^{(n'-n)} \; (\tfrac{1}{2}|\lambda|[x'^2 + y'^2]), \tag{2.22c}$$

valid for $n \leqslant n'$.

The representation properties of composition $\tilde{\mathcal{D}}(g_1)\,\tilde{\mathcal{D}}(g_2) = \tilde{\mathcal{D}}(g_1 g_2)$ may be shown rather tediously to be valid - in fact, it is rather to be used as a proof of the addition theorem

$$\sum_{n'=0}^{\infty} \mathcal{D}^{\lambda}_{nn'}(g_1)\; \mathcal{D}^{\lambda}_{n'n''}(g_2) = \mathcal{D}^{\lambda}_{nn''}(g_1 g_2) \tag{2.23}$$

involving Laguerre polynomials (24, Sects. 1.4 and 5.2). The Lie algebra may be found obtaining derivatives with respect to the group parameters and valuating at the group identity as in (2.13). Again we find the algebra, but given by the half-infinite matrices (1.16) in the linear combinations (1.3a). The defining number operator (1.24b)-(2.21) is in this representation also diagonal, with values $|\lambda|(n + 1/2)$ on the diagonal.

2.12 The bilateral Mellin eigenbasis.

Not all representation bases must come from subgroup of Hamiltonian-type $(\frac{1}{2}\mathbb{P}^2 + V(\mathbb{Q}))$ self-adjoint operators; the representing matrices/kernels may have both continuous and discrete rows and columns. Consider as a non-standard example the operator

$$\mathbb{D}\,\mu^{\lambda}_{\sigma\rho}(y) = \frac{1}{2}\{\mathbb{Q}^{(\lambda)}\mathbb{P}^{(\lambda)} + \mathbb{P}^{(\lambda)}\mathbb{Q}^{(\lambda)}\}\,\mu^{\lambda}_{\sigma\rho}(y) =$$

$$= -i\lambda(y\frac{d}{dy} + \frac{1}{2})\,\mu^{\lambda}_{\sigma\rho}(y) = \rho\,\mu^{\lambda}_{\sigma\rho}(y), \tag{2.24a}$$

$$\mu^{\lambda}_{\sigma\rho}(y) = (2\pi)^{-1/2}\, y_{\sigma}^{-\frac{1}{2} + i\rho/\lambda} \qquad \rho \in R, \quad \sigma = \pm, \tag{2.24b}$$

where we have used the 'cut' power-functions

$$y_+ = \begin{cases} y \;, & y \geqslant 0 \\ 0 \;, & y < 0 \end{cases} \qquad\qquad y_- = \begin{cases} 0 \;, & y > 0 \\ -y \;, & y \leqslant 0. \end{cases} \tag{2.24c}$$

The representation kernels are now 2 × 2 matrices (rows and colums labelled by $\sigma = \pm$) with integral kernel elements. We must consider the four pieces ($\sigma = \pm, \sigma' = \pm$) for $y' \geqslant 0$ and $y' \leqslant 0$ separately. In constructing the kernels through inner products of (2.24b) we obtain integral representations for the confluent hypergeometric functions $M(a,c,z)$ and $U(a,c,z)$ (16, Eqs. 13.2.1 and 13.2.5):

$$\mathcal{D}^{\lambda(\mathcal{D})}_{\sigma\rho,\sigma'\rho'}(g(z',y',z')) = (2\pi)^{-1}\int_{-\infty}^{\infty} dy\, \mu_{\sigma\rho}(y)^{*}\mu_{\sigma'\rho'}(y+y') \times$$

$$\times \exp[i\lambda(z'+x'\;y'/2 + x'y)]$$

$$= (2\pi)^{-1}\exp[i\lambda(z'+x'y'/2]\; K^{\sigma\sigma'}_{\rho\rho'}(x',y'), \tag{2.25a}$$

$$K^{++}_{\rho\rho'}(x',\, y' > 0) = \Gamma(1/2 - i\rho/\lambda)\; y'^{\,i(\rho'-\rho)/\lambda} \times$$

$$\times\, U(1/2 - i\,\rho/\lambda, 1 + i\,(\rho'-\rho)/\lambda,\, -i\lambda x'y'). \tag{2.25b}$$

$$K^{--}_{\rho\rho'}(x',y' > 0) = -\,e^{\pi(\rho-\rho')/\lambda}\, e^{-i\lambda x'y'} K^{++}_{-\rho',-\rho}(-x',\, y' > 0), \tag{2.25c}$$

$$K^{\pm\pm}_{\rho\rho'}(x',y' < 0) = e^{-i\lambda x'y'}\; K^{\pm\pm}_{-\rho',-\rho}(x',\, -y' > 0), \tag{2.25d}$$

$$K^{-+}_{\rho\rho'}(x',\, y' > 0) = -\,e^{\pi(\rho-\rho')/\lambda}\;\Gamma(1/2 + i\,\rho'/\lambda)\;\Gamma(1/2 - i\,\rho/\lambda) \times$$

$$\times\,[\,\Gamma(1 + i\,(\rho'-\rho)/\lambda)]^{-1}\; y'^{\,i(\rho'-\rho)/\lambda} \times$$

$$\times\, M(1/2 - i\,\rho/\lambda,\; 1 + i\,(\rho'-\rho)/\lambda, -i\lambda x'y'), \tag{2.25e}$$

$$K^{+-}_{\rho\rho'}(x', y' < 0) = e^{-\lambda x' y'} K^{-+}_{\rho\rho'}(x', -y' > 0), \tag{2.25f}$$

$$K^{\pm\pm}_{\rho\rho'}(x', y' \gtrless 0) = 0. \tag{2.25g}$$

The 2 × 2 matrix is lower-triangular for $y' > 0$, and upper-triangular for $y' < 0$ the first and second rows and columns are labelled by $\sigma = +$ and $\sigma = -$ respectively). It should be interesting to verify that the group representation composition and indentity properties hold for the integral kernel (2.25). Differentiating with respect to the group parameters should yield the integral kernels for $Q^{\lambda(\mathcal{D})}_{\sigma\rho,}$, $P^{\lambda(\mathcal{D})}_{\sigma\rho,\sigma'\rho'}$, and $I^{\lambda(\mathcal{D})}_{\sigma\rho,\sigma'\rho'} = \lambda\delta_{\sigma\sigma'}\delta(\rho-\rho')$. These will constitute yet another form for the Heisenberg-Weyl algebra such that $1/2(QP+PQ)$ is the number operator $\delta_{\sigma\sigma'}\rho\delta(\rho-\rho')$. Any self-adjoint operator in $L^2(R)$ can be seen thus to have associated to it an eigenbasis which determines a corresponding representation. The first two cases we gave had Q and P for number operators. These determine that the subgroups $g(x,0,0)$ and $g(0,y,0)$, respectively, be represented by purely diagonal integral kernels (c.f. Eqs. (2.11) and (2.15), these are $\delta(q-q')\, e^{ix'q}$ and $\delta(p-p')e^{ipy'}$, and hence the corresponding representations are said to be *reduced* or *classified* according to a subgroup.

The other three cases seen involve nonsubgroup bases, as the corresponding number operator was given by two Schrödinger Hamiltonians (ℓ and h cases) or by the dilatation generator $\mathbb{D}$. In these cases, the representation is given by matrices/kernels which are not diagonal in any subgroup, as suggested by the absence of Dirac δ's in (2.18) and (2.25). They are truly integral kernel representations of the group.

2.13 Unitary irreducible representations.

We now define and verify the unitarity and irreducibility of the representations obtained in (2.11) (Q), (2.14)(P), (2.18)(ℓ),(2.22) (h) and (2.25) ($\mathcal{D}$). In each of these cases we chose a basis for $L^2(R)$ through a self-adjoint operator and the bases are therefore orthonormal. The Heisenberg-Weyl algebra generators are self-adjoint in their Schrödinger realization in $L^2(R)$, and their exponentiated group operators representing (2.1) *unitary*. A unitary operator ($\mathbb{U}(g)f=f_g$ such that $(\mathbb{U}(g)f, (\mathbb{U}(g)h) = (f,h)$ in an orthonormal basis is represented by a unitary matrix or integral kernel:

$$\mathcal{D}^{\lambda}_{\mu\mu'}(g) = [\, \mathcal{D}^{\lambda}_{\mu'\mu}(g^{-1})]^{*}\;, \tag{2.26}$$

where $\mu \in R$ for cases Q, P and ℓ, $\mu \in \{0,1,2,\ldots\}$ for case h and $\mu = (\sigma,\rho)$, $\sigma = \pm$, $\rho \in R$ for case $\mathcal{D}$. For λ real, this is indeed the case, manifestly, for cases P, Q and ℓ. In the latter care has to be taken to define the phase of the parameters in $g^{-1}(x',y'z') = g(-x',-y',-z')$. The matrix for the h-case also satisfies (2.26) with both (2.24b) and (2.24c) required. The last case, $\mathcal{D}$, can also be seen from (2.25) using the pairs (b,d), (c,d) and (f,g) which correspond under the adjunction of the 2×2 matrix. In all of these matrices no block-decomposition occurs and the representations are in fact irreducible. This is to be expected, as $L^2(R)$ itself is irreducible under (2.7).

2.14 Equivalence of representations.

By equivalence of two representations $\underset{\sim}{D}^{(1)}(g)$ and $\underset{\sim}{D}^{(2)}(g)$ of a group G, associated to given number operators $\mathbb{H}^{(1)}$ and $\mathbb{H}^{(2)}$, we mean that there exist (fixed) invertible transformations $\underset{\sim}{C}$ of $L^2(R)$ such that the two representations are mapped into each other as

$$\underset{\sim}{D}^{(2)}(g) = \underset{\sim}{C}\, \underset{\sim}{D}^{(1)}(g)\, \underset{\sim}{C}^{-1} \quad , \quad \forall\, g \in G, \tag{2.27a}$$

$$\underset{\sim}{D}^{(1)}(g) = \| D^{(1)}_{\mu\mu'}(g) \| \, , \quad \underset{\sim}{D}^{(2)}(g) = \| D^{(2)}_{\rho\rho'}(g) \| \, , \quad \underset{\sim}{C} = \| C_{\mu\rho} \| \, , \tag{2.27b}$$

where the set of values which μ can take is the generalized spectrum of $\mathbb{H}^{(1)}$ in $L^2(R)$, and ρ in that of $\mathbb{H}^{(2)}$ in $L^2(R)$:

$$\underset{\sim}{H}^{(2)} = \underset{\sim}{C}\, \underset{\sim}{H}^{(1)}\, \underset{\sim}{C}^{-1}, \quad \underset{\sim}{L}^{(2)} = \underset{\sim}{C}\, \underset{\sim}{L}^{(1)}\, \underset{\sim}{C}^{-1}, \quad \forall\; \underset{\sim}{L} \in \rho\,(a). \tag{2.27c}$$

If $H^{(1)}, H^{(2)} \in a$ then C in (2.27c) is a (possibly exterior) automorphism of the algebra, mapping one subgroup basis into another subgroup basis. An example is provided by the two subgroup representations denoted by Q and P, as we shall see below.

If the $\mathbb{H}^{(\cdot)}$ are self-adjoint and have the same spectrum, $\underset{\sim}{C}$ will be a unitary transformation of $L^2(R)$, i.e. $\underset{\sim}{C}^{-1} = \underset{\sim}{C}^{\dagger}$. This is the for the transformation connecting the Q and P representations -the Fourier transform- which in fact maps $L^2(R)$ unitarily onto itself. There $C^{\lambda}(q,p) = (2\pi)^{-1/2} \exp(ipq/\lambda)$ is the transformation kernel. The Q- and ℓ-representations were defined through eigenbases of the operators $\mathbb{Q}$ in (2.10a) and $\mathbb{H}^{\ell}$ in (2.17a), and have the same spectrum. The transformation kernel $\underset{\sim}{C}$ is then

$$C^{\lambda}_{q,\alpha} = (\chi^{\lambda}_{q}, \Lambda^{\lambda}_{\alpha}) = |\lambda|^{-1/2}\, \Lambda^{\lambda}_{\alpha}(q/\lambda), \tag{2.28}$$

i. e. the Airy transform (14, Sect. 8.5.3), which as expected is unitary in $L^2(R)$. This is a *point* transformation, where to the operator representing Q we may add any function of its canonically conjugate P, as $Q \to Q + f(P)$, while leaving P invariant. Thus

$$J(H) = \frac{1}{2\pi}\oint dq\; p = \frac{1}{\pi}\int_{q_-}^{q_+} dq'[2(H(q,p)-V(q'))]^{1/2} .$$

The angle variable is (26, p. 292) -with a minus sign-

$$\omega(H,T) = -(\partial H/\partial J)T. \tag{2.31b}$$

One can depart slightly from the usual notation: Since $q_\pm$ are the libration endpoints where $V(q_\pm) = H$ (while otherwise $H < V(q)$), the integral (2.31a) is positive. We defined thus

$$J = |\bar{q}|, \qquad \omega = \bar{p}\,\bar{q} / |\bar{q}|, \tag{2.31c}$$

where $\bar{q}$ and $\bar{p}$ is a canonically conjugate pair of variables related to q and p through a canonical transformation.

2.16 Quantum canonical transformations and discrete Hamiltonian spectra.

When Quantum mechanics tries to follow the methods of solution outlined above, the first problem it encounters is that the operators H and T or J and ω *cannot* be the generators -together with I- of a Heisenberg-Weyl algebra, when the spectrum of either of them is lower bound and/or discrete, and I a multiple of the identity operator. Discrete spectra are disallowed by the following contradiction: Let $\phi_m(y)$ be the eigenfunctions of a $\mathbb{Q}^?$ with eigenvalues $m \in Z$. Then, if there exists a conjugate companion $\mathbb{P}^?$ in a Heisenberg-Weyl algebra (1.1), the (m, m') element of the commutation relation would be

$$(\phi_m, [\mathbb{Q}^?, \mathbb{P}^?]\,\phi_{m'}) = (\phi_m, [\mathbb{Q}^?\mathbb{P}^? - \mathbb{P}^?\mathbb{Q}^?]\,\phi_{m'}) =$$

$$= (\mathbb{Q}^{?\dagger}\phi_m, \mathbb{P}^?\phi_{m'}) - (\phi_m, \mathbb{P}^?\mathbb{Q}^?\phi_m) = (m-m')(\phi_m, \mathbb{P}^?\phi_{m'}). \tag{2.32a}$$

But on the other hand,

$$(\phi_m, [\mathbb{Q}^?, \mathbb{P}^?]\,\phi_{m'}) = i\,(\phi_m, \mathbb{I}^?\phi_{m'}) = i\delta_{m,m'}. \tag{2.32b}$$

For $m \neq m'$, $(\phi_m, \mathbb{P}^?\phi_{m'})$ is zero, while for $m = m'$ it is undefined. This is a very old and standard argument, dating back to Jordan in 1927 (27, Postulate D on p. 812, and statement of p. 819, 28 p. 2; 29-31). The same commutation relation also *cannot* be satisfied by any two bounded operators (32, Sect. 6.1.1).

2.17 Quantum canonical transformation to action and angle variables.

In the teeth of the above remarks, Moshinsky and Seligman (33-35) have looked at the quantum mechanical version of the canonical transformation $(q,p) \xrightarrow{C} (\bar{q},\bar{p})$. The first observation is that even in classical mechanics, this transformation is generally not bijective, i. e. the classical phase space motion may be subject to a group of transformations which leave the action-angle variables invariant, as the projection of the Riemann surface for a multivalued function over the complex plane, under exchange of sheets. This *ambiguity group* A is therefore an object which appears already in classical mechanics. Quantizing the system now means replacing q and p for self-adjoint operators $\mathbb{Q}$ and $\mathbb{P}$ in some appropiate representation, say, the Schrödinger representation on $L^2(R)$, while $\bar{q}$ and $\bar{p}$ are replaced by operators $\bar{\mathbb{Q}}$ and $\bar{\mathbb{P}}$ ... in what space? If we have constraining potentials giving rise to a closed classical orbit in phase space and were to propose simply $L^2(R)$, the spectrum of $\bar{\mathbb{Q}}$ would be lower-bound and discrete. In what space may we define $\bar{\mathbb{Q}}$ such that its spectrum be R, as it is for $\mathbb{Q}$? This is of interest since we desire that the quantum canonical transformation be unitary. As we have also the ambiguity group, we may build a space $L^2(R,A) = \sum_{g \in A} L^2_g(R)$ consisting of the ordinary $L^2(R)$ summed with itself once for every element g of the ambiguity group, so as to obtain discrete and perhaps infinite matrices with integral kernel elements. This is the space where $\bar{\mathbb{Q}}$ may act so as to have R for its spectrum, and here the canonical transformation to the original $L^2(R)$ may be bijective and unitary.

The space $L^2(R,A)$ is then classified in a way where linear combinations of the $L^2_g(R)$ spaces are taken so as to build the unitary irreducible representations of the ambiguity group A; in this way one defines an ambiguity *spin*. Some lower-bound discrete spectra allow for a dihedral ambiguity group of rotations by multiples of the period, and inversions. The corresponding ambiguity spin is a pair of numbers: a continuous one $\kappa \in [0,1)$ and a sign σ. The space L^2 is thus written as $\int_0^1 d\kappa \sum_{\sigma=\pm} L^2_{\kappa,\sigma}(R)$.

The operators $\bar{\mathbb{Q}}$ found by Moshinsky and Seligman (33) were such that they are a sum of a Hamiltonian operator with a lower bound and discrete spectrum, *times* the sign σ *plus* the representation index κ. In all, thus, the spectrum of $\bar{\mathbb{Q}}$ is R. Transform kernels are found which represent the above quantum canonical transformation.

2.18 Quantum mechanics on a continuous compact space.

There are other approaches to the problem of making the Hamiltonian of a contraining quantum potential to fit into a Heisenberg-Weyl algebra (3, Refs. 138-154). One of them follows a suggestion by Weyl (36) introducing a *mixed group* $W^\star$ (3, Sect. VI): It is a proper subgroup of W with the composition rule (2.2), and defined through

$$W^{\star} = \{ g\,(x,y,z) \in W \mid x = n_x/M,\ n_x \in Z; \quad y \equiv y \bmod L; \quad z \equiv z \bmod L/2M \}. \tag{2.33}$$

It is a Lie group of transformations on the circle as in (2.7), where x is a discrete group coordinate for the subgroup elements H_Q while the subgroup manifold of H_P is a circle. For the latter, the infinitessimal generator $\mathbb{P} = -i\partial_y$ has, discrete eigenvalues $p = n_p p_0$, $n_p \in Z$, $p_0 = 2\pi/L$. The former group H_Q does *not* have an infinitessimal generator. It has a *finite* generator instead: multiplication by e^{iy}. The eigenvalues of the H_I generator are similarly quantized to $\lambda = n_\lambda \lambda_0, n_\lambda \in Z$, $\lambda_0 = 4\pi M/L = 2\,p_0\,M$. In this model for Quantum mechanics on compact spaces one way work with a properly defined enveloping algebra, implement quantization procedures, classical limits and canonical transformations. Furthermore, as will be brought out in Chapter 3, in connection with the *2+1* Lorentz group, one can define a nonlocal inner product on the circle so that the operator $-id/dy$ have a lower-bound spectrum (37). On the other hand, one does *not* have a self-adjoint 'position' operator, and has in effect quantized on the Lorentz group level.

2.19 Left- and right-invariant Haar measure.

Having gained familiarity with the Heisenberg-Weyl algebra and group, we may state what a noncompact group is, and what its representations are like (20, Sect. 2.3).

Consider a Lie group G with a finite number D of continuous parameters $\{X_k\}_{k=1}^{D}$, $X \in R \subset R^D$, and functions $f(X)$ over G. A positive *Radon* measure is a positive linear form

$$\mu(f) = \int_G d\mu(g) f(g) = \int_G \omega(X) d^D X f(X) \geq 0 \tag{2.34}$$

on the space $C_0^+(G)$ of continuous nonnegative functions f on G with support on a finite-radius sphere. The *left* (resp. *right*) invariant Haar measure is a positive Radon measure which is left (resp. right) invariant under the group action (2.4): $\mu^L(f) = \mu^L(f^L_{g'})$ (*resp.* $\mu^R(f) = \mu^R(f^R_{g'})$), for all $g' \in G$. A theorem may be proven (38, Sect. IV-15): Every Lie group has a *unique* left (and right) invariant Haar measure, up to multiplication by positive constants.

If we fix the multiplicative constant, we may define the *volume* of the group G, vol G, to be the limit of the sequence of values of (2.34) when f is taken as a sequence of characteristic functions over any nested growing sequence of group subsets. This is (2.34) for $f(g) = 1$ when it exists. If the volume of G is a finite number, the group is said to be *compact*. If it is not finite, the group is said to be *noncompact*.

Let $\mu^L(f)$ be a given left Haar measure. We can produce a new left Haar measure through acting on the *right* of the argument with a fixed group element: $\mu^L_{g_0}(f) = \mu^L(f^R_{g_0})$. But since right and left actions commute, i. e. $(f^R_{g''})^L_{g'}(g) \stackrel{\Delta}{=} f(g'^{-1}gg'') = (f^L_{g'})^R_{g''}(g)$, it follows that $\mu^L_{g_0}(f^L_{g'}) = \mu^L((f^R_{g_0})^L_{g'}) = \mu^L(f^R_{g_0}) = \mu^L_{g_0}(f)$, so $\mu^L_{g_0}$ is also a left-invariant measure. By the uniqueness theorem, we conclude that $\mu^L_{g_0}(f) = \Delta(g_0)\mu^L(f)$, i. e. $\mu^L_{g_0}$ may at most differ by a constant $\Delta(g_0)$ from μ^L.

2.20 Unimodular groups.

A *modular* function over G is a positive function $\Delta: G \longrightarrow R^+$ such that $\Delta(gg') = \Delta(g)\Delta(g')$. This implies $\Delta(e)=1$ and $\Delta(g^{-1})=1/\Delta(g)$. The constants $\Delta(g_0)$ seen above are modular functions, as can be verified acting with two elements from the right on the left-invariant measure. If $\Delta(g_0)=1$ for all $g_0 \in G$, then the left-invariant Haar measure is also invariant under right action, so that the left- and right-invariant measures are the same. Such groups are called *unimodular*.

The difference between right and left Haar measures never appears in compact group theory: Every compact group is unimodular. Proof: If G is compact, $f(X)=1$ is in $C^+_0(G)$ and we may normalize μ^L through asking for $\mu^L(1)=1$. The modular function of the group is then $\Delta(g) = \Delta(g)\mu(1) = \mu(1^R_g) = \mu(1) = 1$.

Abelian groups -compact or noncompact- are unimodular, since their left and right actions are the same. Noncompact groups may be non-unimodular, an example of this is the two-parameter solvable group of linear transformations $x \longrightarrow x' = a_1x+a_2$ seen in (39, p. 316). Noncompact groups which *are* unimodular are the following: (a) all abelian groups, (b) all semisimple groups, (c) all connected nilpotent groups -as the Heisenberg-Weyl group, (d) Lie groups for which the range of values of modular functions is compact. (e) direct products of unimodular groups.

2.21 The Haar measure and weight functions.

When the composition functions for the parameters of a group, are known, it is not difficult to build a weight function for a left-invariant Haar measure. Right-invariant Haar measures are very similar and will be given below. We require $\mu^L(f) = \mu^L(f^L_{g_0})$ in (2.34). This implies

$$\mu^L(f^L_{g_0}) = \int_G d\mu^L(g) f(g_0^{-1}g) = \int_{g_0G} d\mu^L(g_0g) f(g) = \mu^L(f). \tag{2.35a}$$

Since $g_0G = G$, we need $d\mu^L(g_0g) = d\mu^L(g)$ for all $g_0 \in G$. If the coordinates of g are X and those of $g'=g_0g$ are X' then the weight function and parameter volume element must satisfy

$$\omega^L(X)d^DX = \omega^L(X')d^DX \simeq \omega^L(X')J\left(\frac{\partial X'(g_0g(X))}{\partial X}\right) d^DX \quad (2.35b)$$

where $J(\partial\cdot/\partial\cdot) = J(g_0,X)$ is the transformation Jacobian. If the group identity is at $X=0$ so that $g(0)=e$, and $\omega^L(0)$ is a fixed number, (2.35b) gives us the appropiate weight function $\omega^L(X')$ at $g'=g_0$:

$$\omega^L(X') = \omega^L(0)\left[J\left(\frac{\partial X'(g_0g(X))}{\partial X}\right)\Big|_{X=0}\right]^{-1}. \quad (2.36a)$$

This is the left-invariant weight function for the parameter volume element at $g'=g_0$, where the parameters are X' and which is invariant under left group translation.

A similar argument for right-invariant integration, measures and Jacobian leads to

$$\omega^R(X') = \omega^R(0)\left[J\left(\frac{\partial X'(g(X)g_0^{-1})}{\partial X}\right)\Big|_{X=0}\right]^{-1} \quad (2.36b)$$

as the appropiate right-invariant weight function at $g'=g_0^{-1}$ where the parameters are X'.

The measure (2.35) is also invariant under the inversion involution $g' \longrightarrow g'^{-1}$ of the group manifold, (20, p. 69, Prop. 3).

The Heisenberg-Weyl group W, being connected and nilponent, is unimodular. Hence, right and left invariant measures are the same. The Jacobian in (2.36) may be computed from the group composition functions (2.2) for $g'(X')=g_0(X_0)g(X)$ and $g''(X'')=g(X)g_0(X_0)^{-1}$,

$$x' = x_0 + x, y'=y_0+ y, z'=z_0 + z+(y_0x - x_0y)/2 \quad (2.37a)$$

$$x'' = x - x_0, y'' = y-y_0, z'' = z - z_0 - (yx_0 - xy_0)/2 . \quad (2.37b)$$

The Jacobians have 1's on the diagonal and are triangular, therefore the determinants are unity. Normalizing $\omega(0) = 1$, the invariant Haar measure for W is

$$d\mu(X) = dx\ dy\ dz. \quad (2.38)$$

The Heisenberg-Weyl group is very close to the abelian R^3 space. The 'twist' which makes the third parameter compose in a non-abelian way, is relatively minor.

2.22 Completeness of a set of representations.

We would like to insist on the distinction between a representation and its subgroup (or nonsubgroup) reduction. Different examples of the latter have been given in the first part of this Chapter, and refer to the row-column classification through the choice of basis for the homogeneous space of the group. Since we saw they are all unitarily equivalent, we can refer to any one fixed classification through a given basis in writing $\mathcal{D}^{\lambda}_{rr'}(g)$. The (in general, collective) indices r and r' are eigenvalues of a (maximal set of mutually commuting) operator(s) in the right- and left-action enveloping algebra of the group, and self-adjoint under the inner product defined by the Haar measure over G (i.e. in a Hilbert space $L^2(G)$). The eigenvalues may be discrete, continuous or mixed, and their range is assumed to resolve degeneracy completely. These representations compose as (2.23) in the discrete case, and as (2.12) in the continuous one. We shall write $\underset{r}{\mathsf{S}} \cdots$ *for* $\sum_r \cdots$ or $\int dr \ldots$. The *representation* index λ (which in general is a collective index) is usually given as eigenvalue(s) of a (maximal set of algebraically independent) right *and* left invariant operator(s) in the centre of the enveloping algebra, self-adjoint in $L^2(G)$. The range of the (generally collective) representation index $\lambda(\lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_N\})$ is a subset $\hat{G}$ of R^N. In benign cases it is a measurable space, and a *Plancherel* measure and weight function $d\rho(\lambda)=\upsilon(\lambda)d\lambda$ exists.

For W, the centre of the enveloping algebras (2.9a) and (2.9b) is $\mathbb{I}^{W(R)}=-i\partial/\partial z = -\mathbb{I}^{W(L)}$, so this generator takes the place of the Casimir operator. The eigenvalue of $-i\partial/\partial z$ provides the representation label $\lambda \in R$.

The difficulty in treating the representations of noncompact groups vis-à-vis the same task for compact groups is the same as that of Fourier integral transforms over Fourier series. The space being noncompact in the former case will allow continuous spectra for certain operators whose eigenvectors are normalizable only in the Dirac sense. Casimir operators may also have *mixed* (continuous and discrete) spectra. In the W case, the difficulties will be handled through Fourier transform techniques. In general one has to invoke rigged Hilbert spaces (20, Chapter 14) and pay close attention to functional analysis arguments (40, 41). The problem for compact group are less, as there all unitary representations are finite dimensional and representations come in discrete series.

2.23 The orthogonality and completeness relations for the unitary irreducible representation matrix elements.

For unimodular groups (with some mild conditions), we can generally assert that the unitary irreducible representation matrices $\mathcal{D}^{\lambda}_{rr'}(g)$ are a generalized orthonormal set of functions over G, under the inner product of $L^2(G)$ given by the invariant Haar integral. We denote the latter through $\underset{g\in G}{\mathsf{S}} = \int_G d\mu(g)\cdots$, in order to include the finite and compact cases in our formulae. We may write

$$(\mathcal{D}^{\lambda}_{rr'}, \mathcal{D}^{\lambda'}_{r''r'''})_G = \underset{g\in G}{\mathsf{S}} [\mathcal{D}^{\lambda}_{rr'}(g)]^* \mathcal{D}^{\lambda'}_{r''r'''}(g)$$

$$= \delta_{\hat{G}}(\lambda,\lambda')\, \delta_{r,r''}\delta_{r',r'''}, \qquad (2.39)$$

where $\delta_{r1,r2}$ means a collective Kronecker delta in discrete indices, (which may be generalized to a Dirac δ's over a continuous pair), $\delta_{\hat{G}}(\lambda,\lambda')$ should play the role of the Kronecker or Dirac δ's for $\lambda,\lambda' \in \hat{G}$, ($\hat{G}$ being endowed with a Plancherel measure and weight function $d\rho(\lambda) = \upsilon(\lambda)d(\lambda)$. Equation (2.39) is not hard to prove through integration by parts with the operators which determine the representation and row/column labels. Most interesting is the statement that the $D^{\lambda}_{rr'}(g)$ are, moreover, *complete* in the $L^2(G)$ Hilbert space with inner product $(\cdot,\cdot)_G$. This means that we may define an inner product in $\hat{G}$ through an integration $\mathbf{S}_{\lambda\in\hat{G}} \cdot\cdot = \int_{\hat{G}} d\rho(\lambda)..,$ which may contain a sum, if the Plancherel measure is a point measure in some domain. We have

$$(D(g_1), D(g_2))_{\hat{G}} = \mathbf{S}_{\lambda\in\hat{G}} \ \mathrm{Tr}\,[D^{\lambda}(g_1)^{\dagger} D^{\lambda}(g_2)]$$

$$= \delta_G(g_1,g_2) \ , \tag{2.40a}$$

where the trace of a product of matrices is

$$\mathrm{Tr}\,(M\,N) = \sum_{rr'} M_{rr'} N_{r'r} \ , \tag{2.40b}$$

For integral kernels an integral over r and r' is used. Equation (2.40a) defines an inner product between matrix or integral kernel functions over $\lambda \in \hat{G}$. The completion of this space of functions with finite norm defines a Hilbert space $L^2(\hat{G})$. The $\delta_G(g_1,g_2)$ is a Kronecker or Dirac delta for discrete or continuous groups: $\delta_G(g_1,g_2)=0$ for $g_1 \neq g_2$, and under sum or integration, it satisfies

$$\mathbf{S}_{g\in G} \ f(g)\delta_G(g,g') = f(g') \tag{2.41a}$$

for any continuous function $f(g)$. Similarly, $\delta_{\hat{G}}(\lambda,\lambda')$ is such that

$$\mathbf{S}_{\lambda\in\hat{G}} \ \hat{F}(\lambda) \ \delta_{\hat{G}}(\lambda,\lambda') = \hat{F}(\lambda') \tag{2.41b}$$

for continuous matrices or integral kernel functions $\hat{F}$ of $\lambda \in \hat{G}$. The defining properties (2.41) imply a relation between weight functions and δ's, as

$$\omega(g(X))\delta_G(g(X),g(X')) = \delta^{\mathcal{D}}(X-X'), \tag{2.42a}$$

$$\upsilon(\lambda)\ \delta_{\hat{G}}(\lambda,\lambda') = \delta(\lambda-\lambda'). \tag{2.42b}$$

In the case of compact groups the Plancherel measure is a point measure and $S_{\lambda\in\hat{G}}\cdots = \sum_\lambda \dim(\lambda)/vol\ G\cdots$ where $\dim(\lambda)$ is the dimension of the λ unitary irreducible representation, and vol G is the volume of the group. The corresponding $\delta_{\hat{G}}$ is thus a Kronecker $\delta_{\hat{G}}(\lambda,\lambda') = \delta_{\lambda\lambda'}$ vol $G/\dim(\lambda)$.

For compact groups, the Peter-Weyl theorem (20, Sect. 7.2, 42) supports the proof of (2.39)-(2.40). For finite groups this can be found in (39, Eqs. (3.143), completeness appears only as (3.178)). The general case of unimodular noncompact groups appears and is referenced to in Barut and Raczka's book (20, Chapter 14).

2.24 The Heisenberg-Weyl case.

We return to our example W and choose for simplicity the Q-eigenbasis where $\mathcal{D}^\lambda_{qq'}(g(x,y,z))$ is given by (2.11). The Haar measure is given by (2.38) with unit weight function and hence δ_G in (2.42a) is an ordinary Dirac δ in x, y and z. We do not yet know the unitary irreducible representation space $\hat{W}$, but since $\lambda \in R$, we want to determine if R is or acts as the full representation space. To this end we have avail to (2.39). If R were not $\hat{W}$, we would not get a Dirac δ in this variable. We perform

$$(\mathcal{D}^\lambda_{qq'},\mathcal{D}^{\lambda'}_{q''q'''})_W = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\ \{\delta(q-q'+\lambda y)\exp i\,[\lambda(z+xy/2)+xq]\}^* \times$$

$$\times\{\delta(q''-q'''+\lambda' y)\ \exp i\,[\lambda'(z+xy/2)+xq'']\}=$$

$$=(|\lambda||\lambda'|)^{-1}\delta((q-q')/\lambda - (q''-q''')/\lambda)\int_{-\infty}^{\infty} dz\ \ \exp i\,[(\lambda-\lambda')z]\ \times$$

$$\times\int_{-\infty}^{\infty} dx\ \ \exp[(-i\,x)((\lambda'-\lambda)(q''-q''')/2\lambda+(q-q''))] = \tag{2.43}$$

$$= 4\pi^2|\lambda|^{-1}\delta(\lambda-\lambda')\delta(q-q'')\delta(q'-q''') = \delta_{\hat{W}}(\lambda,\lambda')\delta(q-q'')\delta(q'-q''').$$

So we do, indeed, obtain a Dirac δ. From (2.42b), the Plancherel measure for W is

$$d\rho(\lambda) = (|\lambda|/4\pi^2)\, d\lambda\,, \qquad \lambda \in \mathbb{R}. \tag{2.44}$$

Completeness is verified as

$$(\mathrm{D}(g_1),\, \mathrm{D}(g_2))_W = (4\pi^2)^{-1} \int_R |\lambda| d\lambda \int_R dq \int_R dq'\, \mathcal{D}^{\lambda}_{qq'}(g_1)^* \mathcal{D}^{\lambda}_{q'q}(g_2)$$

$$= \delta(x_1 - x_2)\,\delta(y_1 - y_2)\,\delta(z_1 - z_2) = \delta_W(g_1, g_2). \tag{2.45}$$

The orthogonality and completeness relations (2.43) and (2.45) are valid *whatever* subgroup or nonsubgroup row/column classification we choose. Hence, orthogonality and completeness relations similar to these follow not only for (2.11), but for (2.15), (2.18), (2.22) and (2.25).

2.25 Harmonic analysis on a group.

Having the complete and orthonormal generalized basis $\mathcal{D}^{\lambda}_{rr'}(g)$ over G allows us to perform *harmonic analysis* over the manifold G, expressing any function within a wide class $f(g)$ over G, through a series or integral over these functions, as

$$f(g) = \mathop{\mathbf{S}}_{\lambda \in \hat{G}} \sum_{r'r'} \hat{f}_{rr'}(\lambda)\, \mathcal{D}^{\lambda}_{rr'}(g)^*, \tag{2.46a}$$

The 'linear combination coefficients' $\hat{f}_{rr'}(\lambda)$ can be obtained through performing the G-inner product of (2.46a) with $\mathcal{D}^{\lambda'}_{r''r'''}(g)$, exchanging $\mathop{\mathbf{S}}_{g}$ with $\mathop{\mathbf{S}}_{\lambda} \sum_{rr'}$ and using (2.39) so as to obtain

$$\hat{f}_{rr'}(\lambda) = \mathop{\mathbf{S}}_{g \in G} f(g)\, \mathcal{D}^{\lambda}_{rr'}(g). \tag{2.46b}$$

We can think of $\hat{F}(\lambda)$ as a matrix- or integral-kernel-valued function on $\hat{G}$ and write (2.46) as

$$f(g) = \mathop{\mathbf{S}}_{\lambda \in G} \mathrm{Tr}\,[\mathrm{D}(g)\hat{\mathrm{F}}\,] = (\mathrm{D}(g^{-1}), \hat{\mathrm{F}})_{\hat{G}}\,, \tag{2.47a}$$

$$\hat{\mathrm{F}}(\lambda) = \mathop{\mathbf{S}}_{g \in G} f(g)\, \mathrm{D}^{\lambda}(g) = (\mathrm{D}^{\lambda^*}, \mathrm{F})_G. \tag{2.47b}$$

Finally, the Parseval relation

$$(F,H)_G = \mathop{\mathbf{S}}_{g\in G} f(g)^* h(g) =$$
$$= \mathop{\mathbf{S}}_{\lambda\in\hat{G}} \sum_{rr'} \hat{f}_{rr'}(\lambda)^* \hat{h}_{rr'}(\lambda) = \mathop{\mathbf{S}}_{\lambda\in\hat{G}} \mathrm{Tr}[\hat{F}^\dagger(\lambda)\hat{h}(\lambda))] = (\hat{F},\hat{H})_{\hat{G}}, \quad (2.48)$$

holds, telling us that the harmonic transform (2.47) is *unitary* between $L^2(G)$ and $L^2(\hat{G})$.

The subject of orthogonal and complete sets of functions over G and $\hat{G}$ spaces extends to all coset spaces $H\backslash G$ or G/H together with right- or left-invariant measures on these, and a corresponding reduction in the row- and column-indices (43). It also extends to other, more general equivalence sets on the group called bilateral classes (44). Rather than delve on the general theory, we shall give some results for the 2+1 Lorentz group in the next Chapter.

CHAPTER 3: A FURTHER EXAMPLE.

In this Chapter we shall present in some detail the case of the semisimple noncompact Lie group of lowest dimension. In the former two Chapters we dealt with the Heisenberg-Weyl group, which was 'as abelian as possible': Only one of the three commutators in the algebra was nonzero. Now we work on the 'least abelian' of three-parameter groups: The 2+1 Lorentz group SO(2,1) and its covering group $\overline{SL(2,R)}$.

The 2+1 Lorentz group has many resemblances as well as definite differences with the three-dimensional rotation group SO(3), which is probably most familiar for physicists who have worked with quantum mechanical systems such as atoms and nuclei. We shall find all irreducible unitary representations of the algebra and covering group. We then realize both the algebra and the group on a coset space related to the Iwasawa decomposition: the circle, and functions and differential operators thereupon. As a final development, we find the in general nonlocal measure defining Hilbert spaces of functions on the circle, for the various representation series.

3.1 The SO(2,1) group and algebra.

Consider a three-dimensional space R^3 with metric (+,-,-), where the distance is $d^2 = x_0^2 - x_1^2 - x_2^2$. This is invariant under the '2+1' Lorentz transformations. These transformations are represented by (pseudo-) rotations around each of the three axes,

$$\exp(i\psi J_1^0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{ch}\,\psi & \mathrm{sh}\,\psi \\ 0 & \mathrm{sh}\,\psi & \mathrm{ch}\,\psi \end{pmatrix} \Leftrightarrow J_1^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad (3.1a)$$

$$\exp(i\chi J_2^0) = \begin{pmatrix} \mathrm{ch}\,\chi & 0 & \mathrm{sh}\,\chi \\ 0 & 1 & 0 \\ \mathrm{sh}\,\chi & 0 & \mathrm{ch}\,\chi \end{pmatrix} \Leftrightarrow J_2^0 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad (3.1b)$$

$$\exp(i\phi J_0^0) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \Leftrightarrow J_0^0 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.1c)$$

We have expressed the rotation around the k^{th} axis as $\exp(i\alpha J_k^0)$ in order to produce a 3×3 representation of the associated Lie algebra. The distance is also invariant under the inversions $x_0 \leftrightarrow -x_0$ and $x_1 \leftrightarrow -x_1$; these compound (in semidirect product) with (3.1). We shall not consider them in what follows, as we are interested primarily in reaching every element with a Lie generator, and they lie in group components not connected to the identity.

From (3.1) we define the 3×3 faithful group representation of the 2+1 Lorentz group as

$$G^{o}(\alpha,\beta,\gamma) = exp\ (i\alpha J_0^{o})\ exp\ (i\beta J_2^{o})\ exp\ (i\gamma J_0^{o})\ , \qquad (3.2a)$$

$$\alpha \equiv \alpha \bmod 2\pi,\ \gamma \equiv \gamma \bmod 2\pi\ ,\quad \beta \in R, \qquad (3.2b)$$

thereby parametrizing it through the *Euler* angles α,β,γ adapted for the nonpositive metric. The matrices have unit determinant while the generators J_k^{o} are all traceless.

The abstract Lorentz group composition law is obtained from the composition law of the matrix representation (3.2) in the same way as done for the Heisenberg-Weyl group in (2.1)-(2.2). The matrices (3.2) are '2+1' pseudo-orthogonal, i. e. orthogonal with respect to the Lorentz metric L as

$$G^{o}\, L\, G^{oT} = L \quad , \qquad L = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (3.3a)$$

We call this group, Special (i.e. of unit determinant) Orthogonal group in 2+1 real dimensions: SO(2,1). The fact that no isochorous space-time inversions are included is usually denoted through a zero subscript, but we shall omit it understanding that we deal with the connected Lie group only.

Correspondingly, the algebra representation matrices are pseudo-skew-symmetric:

$$J_k^{o}\ L = -L\ J_k^{o\dagger}\ , \qquad k=1,\ 2,\ 0. \qquad (3.3b)$$

and satisfy the SO(2,1) algebra give by

$$[J_1,J_2] = -iJ_0,\ [J_2,J_0] = i\, J_1,\ [J_0,J_1] = i\, J_2. \qquad (3.4)$$

Notice the minus sign in the first commutator: The ordinary three-dimensional rotation group algebra so(3) has a plus sign in its stead. one cannot bring so(3) to (3.4), no matter how we redefine generator signs, unless, of course, we introduce i's ($J_1 \to iJ_1$, $J_2 \to iJ_2$); but as we work with real groups, its representation structure would be readically changed.

3.2 The SU(1,1) algebra and group.

We are familiar with Pauli matrices, so we may look for another representation of the Lie algebra so(2,1) in (3.4) through 2×2 matrices. Indeed, we can associate the representation

$$J_1 \to J_1^u = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Leftrightarrow exp(i\psi J_1^u) = \begin{pmatrix} ch\psi/2 & -ish\psi/2 \\ ish\psi/2 & ch\psi/2 \end{pmatrix}, \tag{3.5a}$$

$$J_2 \to J_2^u = \frac{-i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Leftrightarrow exp(i\chi J_2^u) = \begin{pmatrix} ch\chi/2 & sh\chi/2 \\ sh\chi/2 & ch\chi/2 \end{pmatrix}, \tag{3.5b}$$

$$J_0 \to J_0^u = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow exp(i\phi J_0^u) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}. \tag{3.5c}$$

The matrices to the right define a matrix group

$$G^u\,(\bar\alpha,\bar\beta,\bar\gamma) = exp(i\bar\alpha J_0^u)\ exp(i\bar\beta J_2^u\ )\ exp(i\bar\gamma J_0^u)\ , \tag{3.6a}$$

$$\bar\alpha \equiv \bar\alpha \bmod 4\pi, \quad \bar\gamma \equiv \bar\gamma \bmod 2\pi, \quad \bar\beta \in R. \tag{3.6b}$$

In particular notice that

$$G^u(\bar\alpha,\bar\beta,\bar\gamma) = -G^u(\bar\alpha,+2\pi,\bar\beta,\bar\gamma) = -\ G^u(\bar\alpha,\bar\beta,\bar\gamma+2\pi) = G(\bar\alpha+4\pi,\bar\beta,\bar\gamma). \tag{3.6c}$$

The 2×2 matrices (3.6) have unit determinant and the generator matrices in (3.5) are traceless. The abstract group with the composition law obtained from (3.6) constitutes the *special pseudo-unitary group* in 1+1 dimensions SU(1,1):

$$G^u\ \underset{\sim}{\sigma}_3\ G^{u\dagger} = \underset{\sim}{\sigma}_3 \qquad\qquad \underset{\sim}{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.7a}$$

$$J_k^u\ \underset{\sim}{\sigma}_3 = \underset{\sim}{\sigma}_3\ J_k^{u\,\dagger}\ , \qquad k = 1,2,0. \tag{3.7b}$$

The Lie algebra of SU(1,1), su(1,1), is dentical to so (2,1) in (3.4); the groups SU(1,1) and SO(2,1) are not. As in spin angular momentum theory, SU(1,1) *covers* SO(2,1) twice, since between $G^u(\bar\alpha,\bar\beta,\bar\gamma)$ and $G^o(\alpha,\beta,\gamma)$ we can establish a 2:1 mapping given by $G^u(\alpha,\beta,\gamma) \to G^o(\alpha,\beta,\gamma)$ and $G^u(\alpha+2\pi,\beta,\gamma) \to G^o(\alpha,\beta,\gamma)$.

An alternative parametrization of the SU(1,1) group which is sometimes preferable to the Euler angles in (3.6) is found through demanding (3.7a) for any 2×2 complex matrix. This leads to

$$G^u(\eta,\theta) = \begin{pmatrix} \eta & \theta^* \\ \theta & \eta^* \end{pmatrix}, \tag{3.8a}$$

$$|\eta|^2 - |\theta|^2 = 1, \qquad \eta,\theta \in C. \tag{3.8b}$$

In this parametrization, the SU(1,1) manifold is a two-dimensional complex hyperboloid. The ralation between the Euler angle parameters in (3.6) and the parameters in (3.8) favoured by Bargmann (23, Sect. 3b) is easily found (reinterpreted by Sally: 45, p. 3) as

$$\eta = e^{-i(\alpha+\gamma)/2}\,\mathrm{ch}\,\beta/2\,, \qquad \theta = e^{i(\alpha-\gamma)/2}\,\mathrm{sh}\,\beta/2. \tag{3.8c}$$

3.3 The group SL(2,R).

Another parametrization of SU(1,1) is obtained through a similarity transformation of (3.8) as

$$G^S(a,b,c) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} G^u(\eta,\theta) \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} \mathrm{Re}\,\eta - \mathrm{Re}\,\theta & -\mathrm{Im}\,\eta + \mathrm{Im}\,\theta \\ \mathrm{Im}\,\eta + \mathrm{Im}\,\theta & \mathrm{Re}\,\eta + \mathrm{Re}\,\theta \end{pmatrix}, \tag{3.9a}$$

$$ad-bc=1, \qquad a,b,c,d \in R. \tag{3.9b}$$

The corresponding 2×2 algebra representation is

$$J_1^S = \frac{-i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Leftrightarrow \exp\,(i\Psi J_1^S) = \begin{pmatrix} \mathrm{ch}\Psi/2 & \mathrm{sh}\Psi/2 \\ \mathrm{sh}\Psi/2 & \mathrm{ch}\Psi/2 \end{pmatrix}, \tag{3.10a}$$

$$J_2^S = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Leftrightarrow \exp\,(i\chi J_2^S) = \begin{pmatrix} e^{-\chi/2} & 0 \\ 0 & e^{\chi/2} \end{pmatrix}, \tag{3.10b}$$

$$J_0^S = \frac{i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Leftrightarrow \exp\,(i\phi J_0^S) = \begin{pmatrix} \cos\phi/2 & \sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{pmatrix}. \tag{3.10c}$$

In this form, two further group isomorphisms are displayed: Since $G^S(a,b,c)$ is the most general unimodular (unit determinant) 2×2 real matrix, the group represented here is clearly SL(2,R).

The 2×2 unimodular real matrices (3.10) have the further property:

$$G^S\, S\, G^{S^T} = S\,, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.11}$$

where S is the *symplectic* metric matrix. The relation (3.11) defines the two-dimensional real symplectic group Sp(2,R). This group and its 2*N* -dimensional versions Sp(2*N*,R) are important as dynamical groups -rather, algebras- for the *N* -dimensional harmonic oscillator.

3.4 The covering group $\overline{SL(2,R)}$.

We saw $SO(2,1) \overset{1:2}{\approx} SU(1,1) \overset{1:1}{\approx} SL(2,R) \overset{1:1}{\approx} Sp(2,R)$. What are their connectivity properties? A convenient handle is provided by the complex hyperboloid in the η-Θ plane of SU(1,1). Although we need four dimensions, we can look at the surface described by $(\mathrm{Re}\eta)^2+(\mathrm{Im}\eta)^2-(\mathrm{Re}\Theta)^2=1+(\mathrm{Im}\Theta)^2$. For fixed $\mathrm{Im}\,\Theta=0$, the remaining three parameters are constrained to a one-sheeted equilateral hyperboloid with a circular waist in the $(\mathrm{Re}\,\eta, \mathrm{Im}\,\eta)$ plane. The group unit $g^u(1,0)$ is on that waist. As we let $\mathrm{Im}\,\Theta$ range over R, the outside of the hyperboloid fills. The group manifold of $g^u(\eta,\Theta)$ is thus the full Θ complex plane, times the η complex punctured by a round hole of radius $1+|\Theta|^2$. We may surround that hole any number of times describing a path which may *not* be continuously deformed to a vanishing loop. The argument of η increases, but whether or not an increase by 2π is considered to bring us back to the starting point depends on the number of distinct Riemann sheets we provide for $g^u(\eta,\Theta)$ in η.

The simply connected universal covering group of $SU(1,1)\approx \approx SL(2,R) \approx Sp(2,R)$, denoted by $\overline{SL(2,R)}$, is that where the argument of η may take any real value without repeating any group element.

A convenient parametrization for $\overline{SL(2,R)}$ is provided by (3.5)-(3.8) with new parameters as

$$\bar{G}^u(\gamma,\omega) = (|\gamma|^2-1)^{-1/2}\begin{pmatrix} e^{i\omega} & \gamma^* e^{-i\omega} \\ \gamma e^{i\omega} & e^{-i\omega}\end{pmatrix} \tag{3.12a}$$

$$\gamma \in C, \quad |\gamma| < 1, \quad \omega \in R, \tag{3.12b}$$

where

$$\gamma = \Theta/\eta, \quad \omega = \arg \eta . \tag{3.12c}$$

The rotation subgroup $exp\,(i\phi J_0^u)$ in particular, unwinds from the twice-covered circle (3.6b) to the real line.

3.5 Raising, lowering and Casimir algebra elements.

In studying the self-adjoint representations of the Lie algebra sl(2,R) we search for matrices or integral kernels to represent the three algebra generators J_k, $k=0,1,2$ in (3.4). It proves convenient to define the complex linear combinations

$$J_+ = J_1 + i\,J_2, \quad J_- = J_1 - iJ_2, \tag{3.13}$$

which together with J_0 also define the algebra though their commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0 . \tag{3.14}$$

If the J_k are represented by self-adjoint matrices/integral kernels $\underset{\sim}{J}_k = \underset{\sim}{J}_k^\dagger$, then $\underset{\sim}{J}_+^\dagger = \underset{\sim}{J}_-$ and $\underset{\sim}{J}_-^\dagger = \underset{\sim}{J}_+$. They are the raising and lowering operators.

An operator in the enveloping algebra of sl(2,$\mathbb{R}$) and in the centre of the right- and left- acting algebra, is the second-order Casimir operator

$$C = J_1^2 + J_2^2 - J_0^2 = J_\mp J_\pm - J_0^2 \mp J_0 . \tag{3.15}$$

3.6 Eigenbases for a representation.

We follows a classical approach which parallels the standard treatment of angular momentum (c.f. 46, p. 24-26), noting that if we construct eigenfunctions of $\mathbb{J}_0$ and $\mathbb{C}$

$$\mathbb{J}_0 \; \phi_\mu^k = \mu \, \phi_\mu^k , \tag{3.16a}$$

$$\mathbb{C} \; \phi_\mu^k = q \, \phi_\mu^k , \quad q = k(1-k), \tag{3.16b}$$

then the $\mathbb{J}_\pm$ will act as raising and lowering operators:

$$\mathbb{J}_+ \phi_\mu^k = c_{k,\mu}^+ \, \phi_{\mu+1}^k , \tag{3.17a}$$

$$\mathbb{J}_- \phi_\mu^k = c_{k,\mu}^- \, \phi_{\mu-1}^k . \tag{3.17b}$$

The value of k will characterize the representation, while the range of μ will determine its rows.

We make two remarks on μ and k. First, (3.17) tells us that if some ϕ_μ^k is a basis function for some representation, then -unless the proportionality constant $c_{k,\mu}^\pm$ in (3.17) be zero at some point- all other $\phi_{\mu+n}^k$, n integer, will be involved. We can thus write the $\mathbb{J}_0$ -eigenvalue as $\mu = m + \varepsilon$, where m is integer and $\varepsilon \in (-1/2, 1/2]$. If we work with the global representations of SO(2,1), $\mathbb{J}_0$ will allow only for integer μ and hence only $\varepsilon = 0$ is allowed. For SU(1,1) $\approx$ SL(2,$\mathbb{R}$) $\varepsilon = 0$ and $\varepsilon = 1/2$ are allowed, as analogues of vector and spinor representations of SU(2). If it is $\overline{\text{SL}(2,\mathbb{R})}$ we are working with, all ε in (-1/2, 1/2] are appropiate. Second, we have followed Bargmann (23) in writing the Casimir operator eigenvalue $q \in \mathbb{R}$ as $k(1-k)$. Its convenience will be seen below. The number $k(1-k)$ is invariant under $k \leftrightarrow 1 - k$ i. e. an inversion through the $k = 1/2$ point in the complex plane. Also, q is real for k real ($k \geq 1/2$ in order not to double-count), but also for $k = 1/2 + i\rho, \rho \in \mathbb{R}$. We thus have

$$q \leq 1/4 \quad \Leftrightarrow \quad k \in \mathbb{R}. \tag{3.18a}$$

$$q = \frac{1}{4} + \rho^2 \geqslant \frac{1}{4} \quad \Leftrightarrow \quad k = \frac{1}{2} + i\rho, \ \rho \in R. \qquad (3.18b)$$

3.7 Self-adjoint representations.

The third assumption in searching for self-adjoint representations is that $\{\phi^k_{m+\varepsilon}\}_{m\varepsilon Z}$ or a proper irreducible subset thereof, constitute an orthonormal and complete basis under an inner product $(\cdot,\cdot)^{k,\varepsilon}$ which defines some Hilbert space $\mathcal{H}^{k,\varepsilon}$ still to be determined. The generator representation $\mathbb{J}_j$, $j=0,1,2$ will be self-adjoint in that space; it will allow us first to fix the proportionality constants $c^{\pm}_{k\mu}$ in (3.17) as

$$|c^{\pm}_{k,\mu}|^2(\phi^k_{\mu\pm 1}, \phi^k_{\mu\pm 1}) = (\mathbb{J}_{\pm}\phi^k_{\mu}, \mathbb{J}_{\pm}\phi^k_{\mu}) =$$

$$= (\phi^k_{\mu}, \mathbb{J}_{\mp}\mathbb{J}_{\pm}\phi^k_{\mu}) = (\phi^k_{\mu}, [\mathbb{C} + \mathbb{J}_0^2 \pm \mathbb{J}_0]\phi^k_{\mu}) = \qquad (3.19)$$

$$= (q + \mu^2 \pm \mu)\ (\phi^k_{\mu}, \phi^k_{\mu}).$$

As we demand that $(\phi^k_{\mu}, \phi^k_{\mu})$ be unity for all μ in the spectrum of $\mathbb{J}_0$, $|c^{\pm}_{k\mu}|^2$ is obtained as

$$|c^{\pm}_{k,\mu}|^2 = q + \mu^2 \pm \mu = (\mu \pm k)(\mu \mp k \pm 1), \qquad (3.20)$$

which must be a positive quantity. This only allows the absolute value of $c^{\pm}_{k\mu}$ to be determined, but if we denote by $\gamma^{\pm}_{k\mu}$ the phase of $c^{\pm}_{k\mu}$, then (3.17) become

$$\mathbb{J}_{\pm}\underset{\sim}{\phi}^k_{\mu} = c^{\pm}_{k,\mu}\,\underset{\sim}{\phi}^k_{\mu\pm 1}, \qquad (3.21a)$$

$$c^{\pm}_{k\mu} = \gamma^{\pm}_{k\mu}\,[(\mu \pm k)(\mu \mp k \pm 1)]^{1/2}. \qquad (3.21b)$$

Eqs. (3.20) and (3.21) contain the information we need to find the self-adjoint representations of $sl(2,R)$. Non-self-adjoint ones will be commented upon later.

3.8 The continuous representation series.

If both ϕ^k_{μ} and $\phi^k_{\mu\pm 1}$ are to be nonzero, (3.20) must be a positive quantity, i. e. $q > -\mu^2 \mp \mu$. The maximum of $-\mu^2 \mp \mu$ occurs at $\mu_{max} = \mp 1/2$ and has a value of 1/4; so recalling (3.18b) for $q > 1/4$, the representation $k = 1/2 + i\rho, \rho \neq 0$ contains all values of $\mu = m + \varepsilon$, $m \in Z$, for any fixed $\varepsilon \in (-1/2, 1/2]$.

The conditions $q > -\mu^2 \mp \mu = 1/4 - (\mu - \mu_{max})^2$ may be satisfied also for a subset of $\varepsilon \in [1/2, 1/2]$ such that the maximum of the right-hand side does not fall in the μ-content of the representation: If $\mu = m + \varepsilon$, the points falling nearest to the maxima $\mu_{max} = \mp 1/2$ do so at a distance $\mu - \mu_{max} = 1/2 - |\varepsilon|$ from it, so that only $q > |\varepsilon|(1-|\varepsilon|) \geq 0$ is required. For any fixed q in $(0, 1/4]$, thus, we can find a range of ε in $(-1/2, 1/2]$ which allows for unbounded values of μ. Writing $q = k(1-k)$, adding $-1/4$ to both sides of the inequality to complete squares, the condition on ε for fixed real k is given by $|k-1/2| < 1/2 - |\varepsilon|$. In particular, for $k = 0$ all ε but $\varepsilon = 1/2$ allow for the unbounded range of μ.

Bargmann (23, see also 45) considers only SU(1,1) representations: $C_q^{\varepsilon=0}$ the *principal (integer) series* for $1/4 \leq q$, and $C_q^{\varepsilon=1/2}$ the *principal (half-integer) series* for $1/4 < q$. The *exceptional* (or *supplementary*) *series* called $C_q^{\varepsilon=0}$ occurs for $0 < q < 1/4$. These together are called the *continuous series*. We summarize them pictorically in Figs. 1a and 1b.

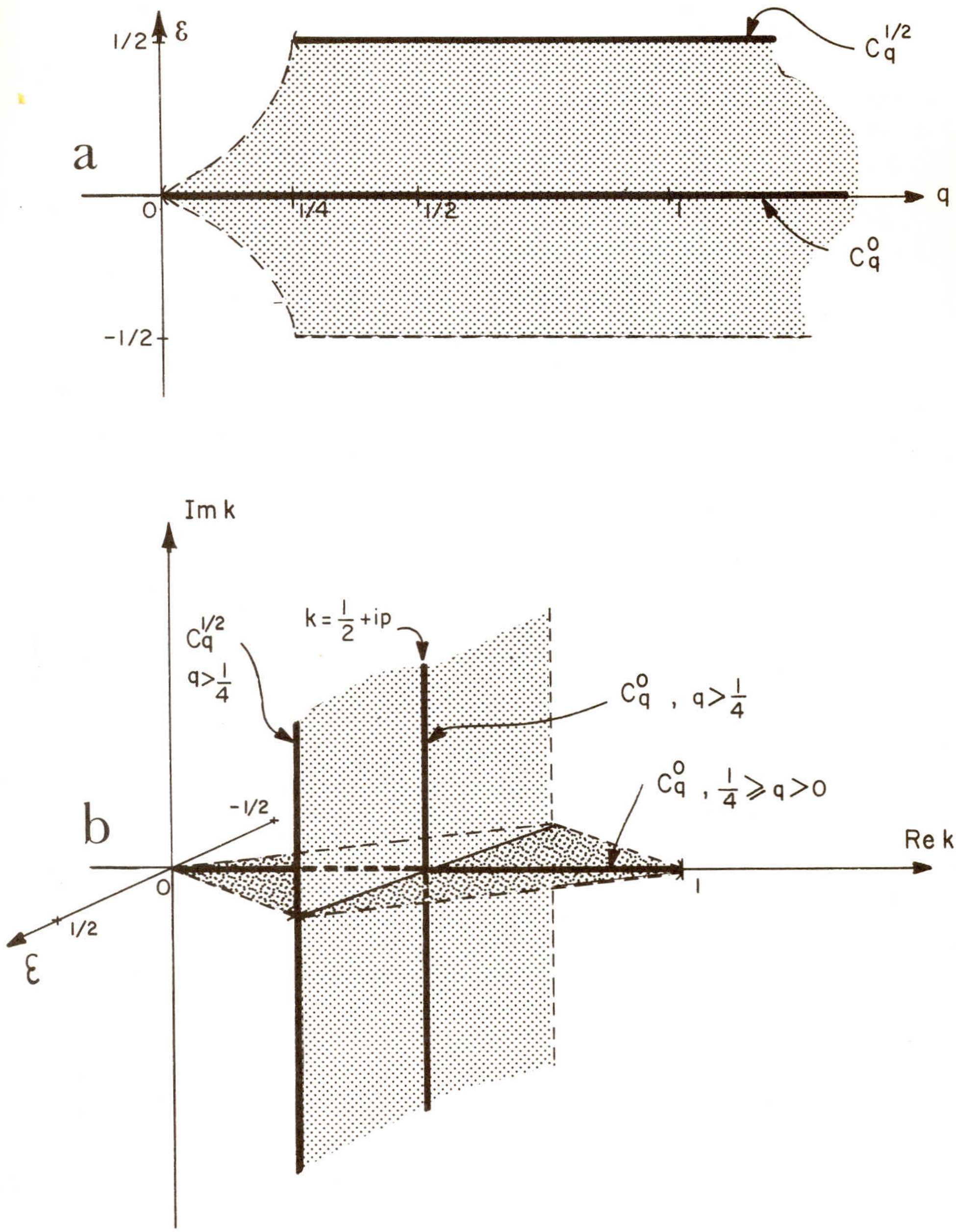
3.9 The 'Discrete' representation series.

We consider next the representations $q \leq 1/4$ for which the positivity of (3.20) is violated for some values of μ, which must be absent from the range of eigenvalues of $\mathbb{J}_0$. From (3.18a), $(\mu \pm k)(\mu \mp k \pm 1)$ has two zeros: $\mu_1^{\pm} = \mp k$ and $\mu_2^{-} = \pm (k-1)$. Hence $|c_{k\mu}^{+}|^2$, as given by (3.20)-(3.21), has a formal negative value for $\mu \in (k-1, -k)$ and similarly $|c_{k\mu}^{-}|^2$ for $\mu \in (k, -k+1)$. No point in the spectrum of $\mathbb{J}_0$ may be in those intervals. See Figs. 2a and 2b.

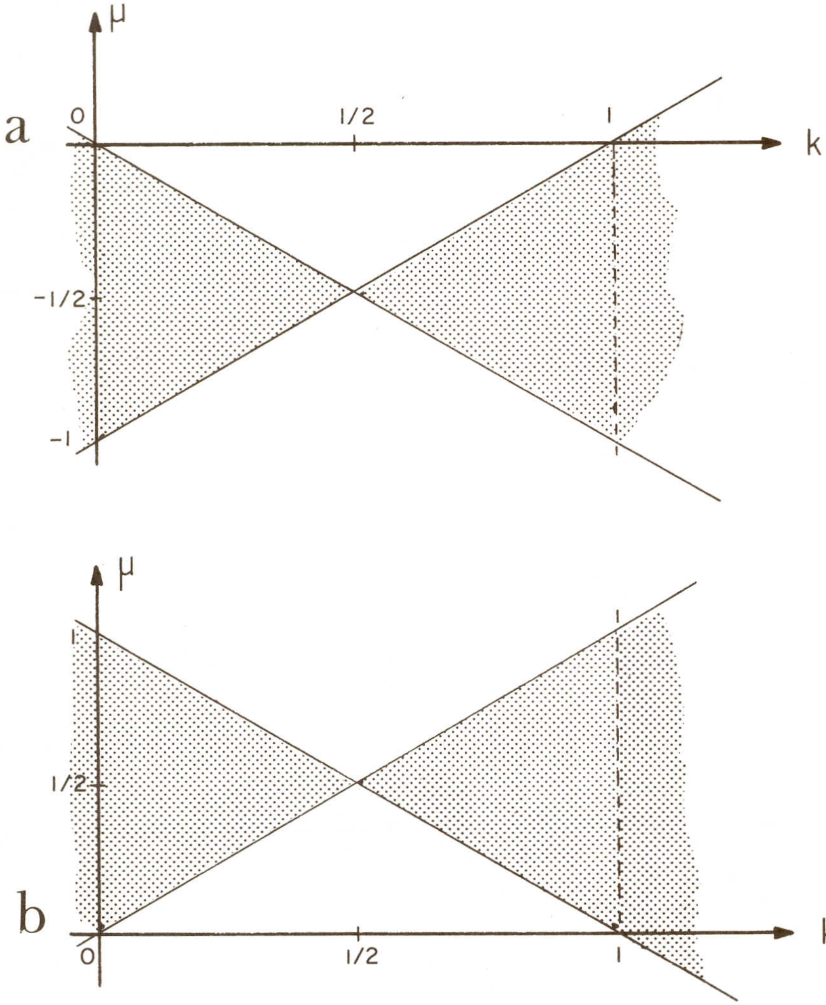
Consider first the case $|k-1/2| > 1/2$ i.e. $k \in (-\infty, 0) \cup (1, \infty)$, where $q < 0$. In those intervals, the distance between $\mu_1^{\pm}$ and $\mu_2^{\pm}$ is larger than 1, so no ε can be chosen such that the sequence $\mu = \varepsilon + m$, $m \in z$ avoids falling in the negative-$|c|^2$ region *unless* it falls on the boundary zeros. The row index μ will then have an upper or lower bound. For, consider what happens when $\mathbb{J}_-$ acts on the 'lowest state' $\phi_{\mu_2^-}^{k>1}$, $\mu_2^- = k$: It gives zero since $c_{k,\mu_1^-}^- = 0$, hence ϕ_{k-1}^k does not belong to the irreducible eigenfunction set of the algebra, which consists then of $\{\phi_{k+m}^{k>1}\}_{m=0}^{\infty}$. The same statement holds for $\phi_{\mu_2^-}^{k<0}$, $\mu_2^- = -k+1$, and only reflects the inversion symmetry under $k^{\mu 2} \leftrightarrow 1-k$. Correspondingly, for $\mathbb{J}_+$, the representation must contain a 'highest state' $\phi_{\mu_2^+}^{k>1}$, $\mu_2^+ = k-1$ which the raising operator will turn to zero. The irreducible eigenfunction set will then be $\{\phi_{k-m-1}^{k>1}\}_{m=0}^{\infty}$, and the analogue statement holds for $\phi_{\mu_1^+}^{k<0}$, $\mu_1^+ = -k$.

These lower- and upper-bound representations were called 'Discrete' series D_k^+ and D_k^- by Bargmann (23). Since he worked with SU(1,1), he allowed only for integer or half-integer values in the spectrum $\{\mu\}$ of $\mathbb{J}_0$, so k was only allowed to be an integer or half-integer. Here, it may be any real number.

When $|k - 1/2| < 1/2$ i.e. $k \in (0,1)$, q covers the interval (0,1/4) twice, and the point $q = 1/4$ once (corresponding to $k = 1/2$). In this interval, the distance between $\mu_1^{\pm}$ and $\mu_2^{\pm}$ is less than 1, so that the sequence $\mu = \varepsilon + m$, $m \in Z$ may 'jump' the forbidden range of μ. This constitutes the exceptional interval of the continuous series seen in the last Section. Again, lower- and upper-bound representations occur when the lowest μ falls on μ_1^- or μ_2^- and



FIGURES 1 The continuous C_q^{ε} irreducible representation series of $sl(2,\mathbb{R})$. (a) as function of real q. (b) as function of complex k for $q=k(1-k)$. Shaded regions correspond to allowed self-adjoint representations. Dashed lines indicate that the region does not include the boundary. The parallel boundaries at $\varepsilon=-1/2$ are to be identified. Heavy lines indicate Bargmann's continuous series C_q^0 and $C_q^{1/2}$.



FIGURES 2 (a) the zeros of $|c^+_{k\mu}|^2$. (b) the zeros of $|c^-_{k\mu}|^2$. Shaded regions correspond to negative values of these quantities which must be excluded from the range of μ.

the highest on μ_1^+ or μ_2^+. We can make use of the inversion symmetry $k \leftrightarrow 1-k$ in order to ascribe the lowest μ of the lower-bound representation to $\mu_1^- = k$, for $k \in (0,1)$ and, correspondingly, the highest μ of the upper-bound representation to $\mu_1^+ = -k$ for $k \in (0,1)$.

3.10 Some isolated points.

The points $k = 1$ and $k = 0$, both mapping on $q = 0$ deserve special attention. The first, $k = 1$, is on par with the neighbouring points along k, it corresponds to $\mathcal{D}_1^{\pm}$ containing $\mu = \pm(1+m), m=0,1,2,\ldots$. The point $k = 0$, however, is unique: It lies in the origin of Figs. 2a and 2b, on zeros of both c_{00}^{+} and c_{00}^{-}. In consequence, $\mathbb{J}_{\pm}\phi_0^0 = 0$, and so the eigenfunction ϕ_0^0 is the basis for the one-dimensional trivial representation of the algebra by zero. As a representation for the group, ϕ_0^0 constitutes the trivial one-dimensional unitary representation.

In summary, the lower- and upper-bound self-adjoint representations of $sl(2,R)$ are $\mathcal{D}_k^{\pm}$, $k > 0$ containing the eigenfunctions ϕ_μ^k of $\mathbb{J}_0$ and $\mathbb{C}$ for $\mu = \pm(k \mp m)$, $m=0,1,2,\ldots$. If $k \in (0,1]$ we can write $\mu = \varepsilon \pm m$ as for the continuous series, but keeping the congruence interval of ε to be $(-1/2,1/2]$. The discrete series provide the boundaries of the open regions of the exceptional interval in Figs. 1a and 1b. In particular, the C_q^0 series convers the interval $q > 0$: The point $q = 0$ belongs to the discrete series $\mathcal{D}_1^{\pm}$ and to the trivial $\mathcal{D}_0^{\pm}$. Similarly, the $C_q^{1/2}$ series covers the interval $q > 1/4$, while the point $q = 1/4$ belongs to $\mathcal{D}_{1/2}^{\pm}$.

For $0 < q < 1/4$ both the exceptional continuous and discrete series coexist. Their basis functions have the same eigenvalue under $\mathbb{C}$ but different spectra under $\mathbb{J}_0$. This underlines the need of having two different Hilbert spaces to accomodate them.

In order to express these facts in the manner of Figs. 1a and 1b, we may let ε range outside the interval $(-1/2,1/2]$, so that the μ-content of $\mathcal{D}_k^{\pm}$ be $\mu = \varepsilon \pm m$, $m = 0,1,2,\ldots$ (i. e. $\varepsilon = \pm k$). We may then, superpose them as in Fig. 3. In order to display the representations clearly in the k-plane, we refer the reader to Figs. 4, where we mark those representations of so(2,1) which contain the eigenvalues $\mu = \varepsilon + m$, i. e. plotting vector representations of SO(2,1) ($\varepsilon = 0$), spinor representations ($\varepsilon = 1/2$), four-fold covering representations ($\varepsilon = 1/4$), etc.

Having classified all self-adjoint representations of the algebra $sl(2,R)$ in terms of eigenbases of the Casimir operator $\mathbb{C}$ labelled by its eigenvalues $q = k(1-k)$, and having specified their representation content with respect to $\mathbb{J}_0$, we can give the generator matrix elements

$$J_j^{k,\varepsilon} = \| (J_j)_{m\,m'}^{k,\varepsilon} \|, \quad (J_j)_{m,m'}^{k\varepsilon} = (\phi_{m+\varepsilon}^k, \mathbb{J}_j\, \phi_{m'+\varepsilon}^k), \; j=1,2,0, \text{ or } \pm 0, \tag{3.22}$$

as

$$(J_0)_{m\,m'}^{k,\varepsilon} = \delta_{m\,m'}(\varepsilon + m) = \delta_{m\,m'}\, \mu(m, \varepsilon), \tag{3.23a}$$

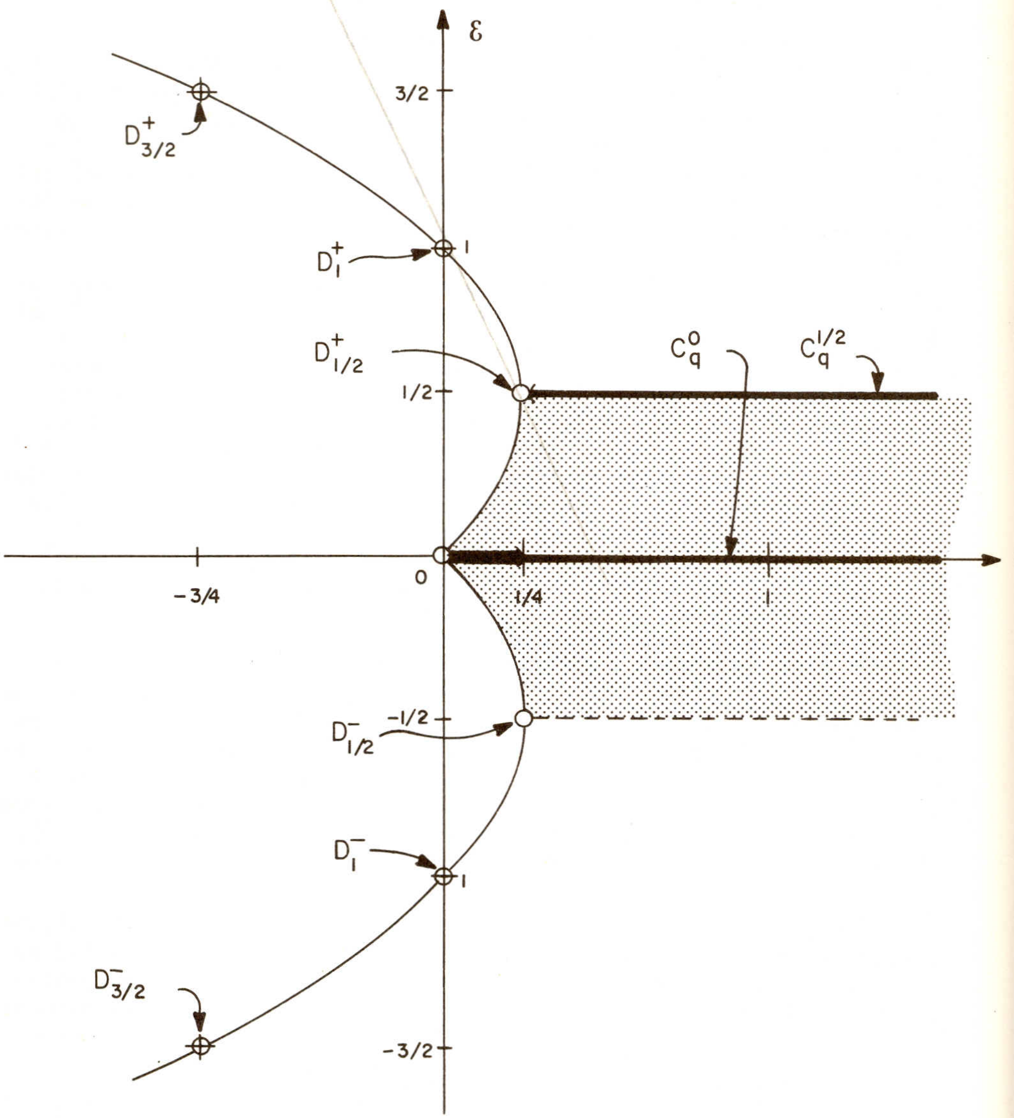
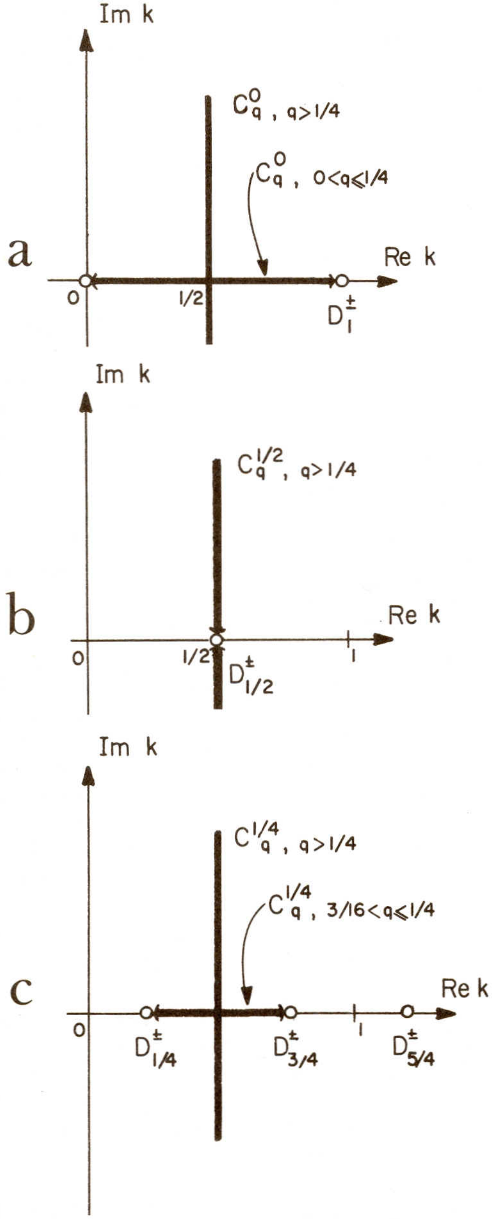


FIGURE 3. Discrete $\mathcal{D}^{\pm}_{k}$ and continuous $\mathcal{C}^{\varepsilon}_{q}$ irreducible representation series.



FIGURES 4. (a) vector, (b) spinor, (c) four-fold-valued irreducible representations of SO(2,1), in the k-plane.

$$(J_+)^{k,\varepsilon}_{m\,m'} = \delta_{m+1,m'}[(m'+\varepsilon+k)(m'+\varepsilon-k+1)]^{1/2} = \delta_{m+1,m'}c^+_{k,\mu(m'\varepsilon)},$$

(3.23b)

$$(J_-)^{k,\varepsilon}_{m\,m'} = \delta_{m-1,m'}[(m'+\varepsilon-k)(m'+\varepsilon+k-1)]^{1/2} = \delta_{m-1,m'}c^-_{k,\mu(m',\varepsilon)}\ .$$

(3.23c)

We may give a résumé of the ranges of the various indices as

$$C^\varepsilon_q:\ m,m' \in Z,\ k \in \{\tfrac{1}{2}+i\rho, \rho \in R\} \cup \{(0,1), |k-\tfrac{1}{2}| < \tfrac{1}{2} - |\varepsilon|\}, \varepsilon \in (-\tfrac{1}{2},\tfrac{1}{2}],$$

(3.24a)

$$D^+_k:\ m,m' \in \{0,1,2,\ldots\},\ k \in (0,\infty),\ \varepsilon = k\ , \qquad (3.24b)$$

$$D^-_k:\ m,m' \in \{0,-1,-2,\ldots\},\ k \in (0,\infty),\ \varepsilon = k. \qquad (3.24c)$$

These are infinite matrices which follow the commutation relations (3.14) and for which the Casimir operator is represented by a multiple q of the unit matrix.

3.11 "Other" representations.

Some words about non-self-adjoint representations. The self-adjointness of (3.22)-(3.24) stemmed out of asking for $q+\mu^2\pm\mu$ in (3.20) to be positive, which in turn came from the $\mathbb{J}_+{}^\dagger = \mathbb{J}_-$ requirement in the second equality of (3.19). If we work only with the raising and lowering action of $\mathbb{J}_\pm$ on some set of vector components as given in (3.21), accepting imaginary $c^\pm_{k\mu}$'s, then we may let k be complex. Choosing any given complex μ as a starting point, we obtain all other $\mu+n$, $n \in Z$ unless we meet a zero coefficient, in which case indecomposability will occur.

Suppose we start with a negative integer k, with ϕ^k_0. We may apply $\mathbb{J}_\pm$ to obtain $\phi^k_{\pm1}, \phi^k_{\pm2}, \ldots$. Figures 2a and 2b tell us that applying $\mathbb{J}_\pm$ a sufficient number of times will take us to $\mu^\pm_1 = \mp k = \pm|k|$, but then as $c^\pm_{k,\pm|k|} = 0$, $\mathbb{J}_\pm \phi^k_{\pm|k|} = 0$. We have thus a $(2|k|+1)$-dimensional non-self-adjoint representation of sl(2,R). The elements of the algebra will be represented as follows: J_0 will be diagonal, real and hence self-adjoint, while $J_+{}^\dagger = -J_-$ so J_1 and J_2 will be skew-adjoint and i times their so(3) analogues. The same procedure applies for any integer or half-integer $k=\pm|k|$, as there the upper and lower bounds given by the zeroes of $c^\pm_{k\mu}$ are separated by integers. Moreover, the offending factors which produce the zero barriers may be removed by a simple norm redefinition of the basis elements ϕ^k_μ, as they are no longer subject to any orthonormality conditions. We may thus build representations with 'one-way' barriers, through which one raise, but not lower μ, or vice-versa. This implies a structure of the type (1.17) at each of two barriers, leading to indecomposable representations with the structures

$$\begin{pmatrix} X & X & X \\ 0 & X & X \\ 0 & 0 & X \end{pmatrix}, \begin{pmatrix} X & 0 & 0 \\ X & X & 0 \\ X & X & X \end{pmatrix}, \begin{pmatrix} X & X & 0 \\ 0 & X & 0 \\ 0 & X & X \end{pmatrix}, \text{and} \begin{pmatrix} X & 0 & 0 \\ X & X & X \\ 0 & 0 & X \end{pmatrix}.$$

Indecomposable representations of so(2,1) have been described by Chacón, Levi and Moshinsky (47), and those of arbitrary semisimple groups by Gruber and Klimyk (48).

3.12 Subgroups and coset spaces.

We have not yet given any *realization* of the so(2,1) algebra generators (3.4)-(3.14) as differential operators on a manifold, nor have we provided the functions ϕ_μ^k or specified the Hilbert spaces for which these will provide complete and orthonormal bases. A general way of providing these is in following the procedure of Chapter 2, namely we let the group act on functions of coset spaces of the group and thus find the infinitesimal generators as differential operators on the coset manifolds. In order to construct these, we should know all nonequivalent subgroups of $\overline{SL(2,R)}$ and preferably divide by the largest proper subgroup so that the coset space have as low dimension as possible, without loosing information. For su(1,1) $\approx$ sl(2,R) this is relatively simple and the results in the 2×2 representation (3.9)-(3.10), and up to equivalence $g \cdot g^{-1}$ are as follows:

a) The *elliptic* subalgebra (subgroup) E

$$J_0 \to J_0^S = \frac{i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \exp(i\phi J_0^s) = \begin{pmatrix} \cos\phi/2 & \sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{pmatrix} \in E \qquad (3.25a)$$

where the parameter ϕ ranges over $[0,4\pi)$ for SU(1,1)$\approx$ SL(2,R), and may be identified through (3.9) and (3.12) with the parameter ω for $\overline{SL(2,R)}$, ranging over the full real line.

b) The *hyperbolic* subalgebra (subgroup) H

$$J_2 \to J_2^S = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \exp(i\chi J_2^S) = \begin{pmatrix} e^{-\chi/2} & 0 \\ 0 & e^{\chi/2} \end{pmatrix} \in H, \ \chi \in R. \qquad (3.25b)$$

c) The *parabolic* subalgebra (subgroup) P

$$J_0 + J_1 \to J_0^S + J_1^S = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \quad \exp(i\xi[J_0^S + J_1^S]) = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \in P, \ \xi \in R. \qquad (3.25c)$$

d) There is a single two-parameter *solvable* subalgebra (subgroups) S obtained from H and P generated by J_2 and $J_0 + J_1$, with group elements

$$s(\chi,\xi) = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-\chi/2} & 0 \\ 0 & e^{\chi/2} \end{pmatrix} = \begin{pmatrix} e^{-\chi/2} & \xi\, e^{\chi/2} \\ 0 & e^{\chi/2} \end{pmatrix} \in S. \qquad (3.26)$$

We can now select among a number of homogeneous spaces $G_1\backslash G$ where G is SO(2,1) or SL(2,R), and G_1 may be any of (3.25)-(3.26). The $S\backslash$ SL(2,R) coset space is a one-dimensional manifold, the covering of the circle S_1. It turns out that this manifold by itself is too 'small' to contain all representations, although we shall come back to it below with a new interpretation.

Another most natural choice for coset space is $E \backslash$ SL(2,R), as the ensuing description closely resembles the description of the rotation group SO(3) through functions on the sphere S_2 = SO(2) \ SO(3), and the Euler angle decomposition of SO(2,1) in (3.2) is convenient for its description. The problem with this coset space is that although we do indeed get the expression for the generators $\mathbb{J}_k$, $k=0,1,2$ as in (2.8)), as differential operators of the first degree in β and γ, the Casimir operator will be a second-order differential operator resembling the angular part of the Laplacian (46, Eqs. (III-6)-(III-8), except for its eigenvalues. These functions are rather tedious to work with in integrations. A similar outcome awaits H\SL(2,R) (49, the functions $\mathfrak{P}_{0n}(ch\beta)e^{in\alpha}$ in Chapter VI, Eqs. 3.2(6) and 4.5(9')).

3.13 The Iwasawa decomposition.

The simplest approach by far is to use the coset space $P\backslash$SL(2,R) with a particularly fortunate choice of group parameters. This is given by the *Iwasawa* decomposition of $g \in$ SL(2,R)

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \exp(-\chi/2) & 0 \\ 0 & \exp(\chi/2) \end{pmatrix}\begin{pmatrix} \cos\phi/2 & \sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi/2\; e^{-\chi/2} - \xi \sin\phi/2\; e^{\chi/2} & \sin\phi/2\; e^{-\chi/2} + \xi \cos\phi/2\; e^{\chi/2} \\ -\sin\phi/2\; e^{\chi/2} & \cos\phi/2\; e^{\chi/2} \end{pmatrix}, \qquad (3.27a)$$

$$\tan\frac{\phi}{2} = -\frac{c}{d},\quad e^{\chi} = c^2+d^2,\quad \xi = \frac{b}{d} + \frac{c/d}{c^2+d^2}. \qquad (3.27b)$$

The Iwasawa decomposition of a noncompact group (20, Sect. 1.6.C) or algebra (50) expresses an arbitrary group element $g \in G$ as $g=nak$ where $k \in K$, the maximal compact subgroup -in this case SO(2) generated by J_0 - times $a \in A$, a maximal abelian subgroup, times $n \in N$, a nilpotent subgroup. The product NA is the solvable group in (3.26).

Functions on the coset space $M=P\backslash$SL(2,R) are two-variable functions $f(\chi,\phi)$ whose transformation properties under right action of the group [c. f. (2.4a)] by some element $g_0 \in$ SL(2,R) is

$$f(\chi,\phi) \xrightarrow{g_0\ (\mathbb{R})} f(\chi'(\chi,\phi,g_0),\phi'(\chi,\phi,g_0)) \ , \qquad (3.28a)$$

and can be obtained through multiplying (3.27a) on the right by such a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{g_0} \begin{pmatrix} a'b' \\ c'd' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} aa_0+bc_0 & ab_0+bd_0 \\ ca_0+dc_0 & cb_0+dd_0 \end{pmatrix}. \qquad (3.28b)$$

Making use of (3.27b) we find that the coset parameters transform as

$$\tan\frac{\phi}{2} \to \tan\frac{\phi'}{2} = -\frac{ca_0+dc_0}{cb_0+dd_0} = \frac{a_0\tan\phi/2 - c_0}{d_0 - b_0\tan\phi/2} \ , \qquad (3.29a)$$

$$e^{\chi} \to e^{\chi'} = (ca_0+dc_0)^2 + (cb_0+dd_0)^2$$

$$= e^{\chi}\cos^2(\tfrac{\phi}{2})[(a_0\tan\tfrac{\phi}{2} - c_0)^2 + (d_0-b_0\tan\tfrac{\phi}{2})^2] \ . \qquad (3.29b)$$

3.14 sl(2,$\mathbb{R}$) algebras of differential operators.

Global transformations will be futher analyzed below; here we are interested in infinitessimal transformations which in 3 × 3 matrix form (3.10) appear as

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = 1 + i\,[\delta_1 J_1^S + \delta_2 J_2^S + \delta_0 J_0^S] + o(\delta^2) = \begin{pmatrix} 1-\frac{1}{2}\delta_2 & \frac{1}{2}(\delta_1+\delta_0) \\ \frac{1}{2}(\delta_1-\delta_0) & 1+\frac{1}{2}\delta_2 \end{pmatrix}. \qquad (3.30a)$$

On a dense subspace of the space of functions (3.28a) on the coset space $M = P \setminus SL(2,\mathbb{R})$, they are given by

$$f(\chi',\phi') = [1+i(\delta_1\mathbb{J}_1^M + \delta_2\mathbb{J}_2^M + \delta_0\mathbb{J}_0^M)+o(\delta^2)]f(\chi,\phi), \qquad (3.30b)$$

where $\mathbb{J}_k^M$, $k=0,1,2$ will be first-order differential operators in χ and ϕ. Replacement of (3.30a) in (3.29), Taylor expansion and collection of δ_k's allows us to find them:

$$\mathbb{J}_1^M = i(\cos\phi\,\partial_\phi + \sin\phi\,\partial_\chi), \qquad (3.31a)$$

$$\mathbb{J}_2^M = i(\sin\phi\,\partial_\phi - \cos\phi\,\partial_\chi), \qquad (3.31b)$$

$$\mathbb{J}_0^M = -i\partial_\phi \quad . \tag{3.31c}$$

They satisfy the commutation relations (3.4). From here we can find the raising and lowering operators (3.13) as

$$\mathbb{J}_+^M = \mathbb{J}_1^M + i\mathbb{J}_2^M = i\, e^{i\phi}\,(\partial_\phi - i\partial_\chi), \tag{3.32a}$$

$$\mathbb{J}_-^M = \mathbb{J}_1^M - i\mathbb{J}_2^M = i\, e^{-i\phi}(\partial_\phi + i\partial_\chi). \tag{3.32b}$$

Most important, we find the Casimir operator (3.14) in this realization to be

$$\mathbb{C}^M = \mathbb{J}_1^{M^2} + \mathbb{J}_2^{M^2} - \mathbb{J}_0^{M^2} = -\,\partial_\chi(1+\partial_\chi). \tag{3.33}$$

One can see the special feature of the Iwasawa decomposition (3.27a) and the parabolic coset space $M=P\setminus SL(2,R)$: That $\mathbb{J}_0^M$ is a differential operator in ϕ and $\mathbb{C}^M$ in χ_M. Functions on M belonging to a given eigenvalue $k(1-k)$ under $\mathbb{C}^M$ can be characterized through

$$f_k\,(\chi,\phi) = e^{-k\chi}\, f_k(\phi). \tag{3.34}$$

On these spaces of functions, ∂_χ may be replaced by $-k$, so as to obtain from (3.31) and (3.32),

$$\mathbb{J}_\pm^{(k)} = i\, e^{\pm i\phi}\,(\partial_\phi \pm ik), \tag{3.35a}$$

$$\mathbb{J}_0^{(k)} = -i\partial_\phi\ . \tag{3.35b}$$

This is the sl(2,R) algebra realization employed by Bargmann (23). Equation (3.34) will be used below to examine the global action of SL(2,R) on this space of functions.

3.15 Basis functions on the circle.

In Bargmann's realization, the normalized eigenfunctions ϕ_μ^k in Eq. (3.16) are proportional to the exponential functions $e^{i\mu\phi}$. The eigenvalue μ we saw, must range in integer steps $\mu=\varepsilon\pm m, m=0,1,2,\ldots$ passing through ε (for C_q^ε) or starting from a value $\mu=\varepsilon$, for $\mathcal{D}_k^\pm$, as specified in Eqs. (3.24), so that $e^{i\mu\phi} = e^{i\varepsilon\phi}e^{im\phi}$. We let ϕ range on the unit circle S_1 (i.e. $\phi\equiv\phi$ mod2π) and write the (possibly multivalued) function $f(\phi)$ on a space associated with k and ε as

$$f(\phi) = e^{i\varepsilon\phi} \sum_n{}^{(k)}\ f_n\, e^{in\phi}\quad , \tag{3.36a}$$

where we are indicating the sum over the proper μ-set as

$$\sum_{n}^{(k)} = \sum_{n=-\infty}^{\infty} [\text{ for } C^{\varepsilon}_{k(1-k)}], = \sum_{n=0}^{\pm\infty} [\text{ for } D^{\pm}_{k}, \text{ with } \varepsilon = \pm k]. \quad (3.36b)$$

The possible multivaluation will not be a problem when calculating inner products - with a measure still to be found- , as any sesquilinear inner product involving $(e^{i\mu_1\phi})^*$ and $e^{i\mu_2\phi}$ with $\mu_1=\varepsilon+m_1$ and $\mu_2=\varepsilon+m_2$ will cancel out the $e^{i\varepsilon\phi}$ factors, or otherwise render them innocuous.

We have not yet written down the *normalization* constants for the eigenfunctions. If we assume that $L^2(S_1)$ is the proper space, these constants will be $(2\pi)^{-1/2}$ for all m. But we may have other spaces. In general their normalization will be prescribed by the condition (3.19)-(3.21).

We may fix the normalization of $\phi^k_{\mu_0}$ for $\mu_0=\varepsilon$ and from this obtain all other $\phi^k_{\varepsilon+m}$, with ε and m ranging as in (3.36), through ascending or descending by means of $\mathbb{J}^{(k)}_{\pm}$:

$$(\mathbb{J}^{(k)}_{\pm})^m \phi^k_{\varepsilon}(\phi) = c^{\pm}_{k\varepsilon} (\mathbb{J}^{(k)}_{\pm})^{m-1} \phi^k_{\varepsilon\pm1}(\phi) =$$

$$= \left[\prod_{n=0}^{m-1} c^{\pm}_{k,\varepsilon+n}\right] \phi^k_{\varepsilon\pm m}(\phi). \quad (3.37a)$$

We have the realization (3.35a) for $\mathbb{J}^{(k)}_{\pm}$ and $\phi^k_{\varepsilon}(\phi)=K^k_{\varepsilon}\, e^{i\varepsilon\phi}$, hence

$$\begin{aligned}
\phi^k_{\varepsilon\pm m}(\phi) &= K^k_{\varepsilon}\left[\prod_{n=0}^{m-1} c^{\pm}_{k,\varepsilon\pm n}\right]^{-1} [ie^{\pm i\phi}(\partial_{\phi}\pm ik)]^m e^{i\varepsilon\phi} \\
&= K^k_{\varepsilon}\left[\prod_{n=0}^{m-1} c^{\pm}_{k,\varepsilon\pm n}\right]^{-1} (-\varepsilon\mp k)[ie^{\pm i\phi}(\partial_{\phi}\pm ik)]^{m-1} e^{i(\varepsilon\pm1)\phi} \\
&= K^k_{\varepsilon}\left[\prod_{n=0}^{m-1} \frac{-\varepsilon\mp k\mp n}{c^{\pm}_{k,\varepsilon\pm n}}\right] e^{i(\varepsilon\pm m)\phi} \qquad (3.37b) \\
&= K^k_{\varepsilon}\, \eta^k_{\varepsilon\pm m}\left[\prod_{n=0}^{m-1}\left|\frac{\varepsilon\pm n\pm k}{\varepsilon\pm n\pm(1-k)}\right|\right]^{1/2} e^{i(\varepsilon\pm m)\phi} ,
\end{aligned}$$

where $\eta^k_{\varepsilon\pm m}$ is the product of the inverse of the phases $\gamma^{\pm}_{k\mu}$ of the $c^{\pm}_{k\mu}$ in (3.21) for $\mu=\varepsilon,\varepsilon\pm1,\ldots,\varepsilon\pm(n-1)$.

We shall not need a definite phase convention in these notes. It should only be noted that phases *can* be arranged so that the $\gamma^{\pm}_{k\mu}=1$, that is, that the raising and lowering operators have purely positive matrix matrix elements. This is the phase definition followed in (23, Sect. 6d) and known as Bargmann's convention.

We use the Pochhammer product symbol

$$\prod_{n=0}^{m-1}(a+n) = a(a+1)\cdots(a+m-1) = (a)_m = \Gamma(a+m)/\Gamma(a), \qquad (3.37c)$$

$$\prod_{n=0}^{m-1}(b-n) = b(b-m+1)_m = \Gamma(b+1)/\Gamma(b+1-m), \qquad (3.37d)$$

so as to write (3.37b) in the form

$$\phi^{k}_{\varepsilon\pm m}(\phi) = K^{k}_{\varepsilon} n^{k}_{\varepsilon\pm m}\ \sigma^{k,\varepsilon}_{\pm m}\ e^{i(\varepsilon\pm m)\phi} \qquad (3.38a)$$

$$\sigma^{k\varepsilon}_{\pm m} = \left|\frac{(k\pm\varepsilon)_m}{(1-k\pm\varepsilon)_m}\right|^{1/2} = \left|\frac{\Gamma(\varepsilon+1-k)}{\Gamma(\varepsilon+k)}\ \frac{\Gamma(\varepsilon+k\pm m)}{\Gamma(\varepsilon+1-k\pm m)}\right|^{1/2} = \left|\frac{\Gamma(1-k\pm\varepsilon)}{\Gamma(k\pm\varepsilon)}\ \frac{\Gamma(k\pm\varepsilon+m)}{\Gamma(1-k\pm\varepsilon+m)}\right|^{1/2}.$$

(3.38b)

We now examine the characteristics of the $\sigma^{k\varepsilon}_{\pm m}$ square root factor for the various representation series. We refer to (3.24)-(3.36b) for notation and index ranges. For the principal series C^{ε}_{q} in the nonexceptional interval, $k=1/2+i\rho$, $\rho\in R$ and hence the ratio of functions of the form $\gamma(1-k)/\gamma(k)=\gamma(1/2-i\rho)/\gamma(1/2+i\rho)$ has absolute value unity.

Thus

$$C^{\varepsilon}_{q},\ q > \frac{1}{4},\quad \sigma^{k,\varepsilon}_{\pm m} = 1 \qquad \text{(continuous nonexceptional).} \qquad (3.39a)$$

For the exceptional interval $k\in(0,1)$, on the other hand, for $k \leqslant 1/2$ we have $(k+\alpha)/(1-k+\alpha) \leqslant 1$. The ratio of Γ-functions becomes a product of such factors for $\alpha=\pm\varepsilon,\pm\varepsilon+1,\ldots,\pm\varepsilon+m-1$, so that

$$C^{\varepsilon}_{q=k(1-k)},\quad k\in\begin{cases}(0,1/2)\\(1/2,1)\end{cases},\quad \sigma^{k\varepsilon}_{\pm m} \leqslant 1 \qquad \text{(continuous exceptional).} \qquad (3.39b)$$

Finally, for the discrete series $\mathcal{D}^{\pm}_{k}$, where $\varepsilon=\pm k$, we may write explicitly

$$\mathcal{D}^{\pm}_{k},\quad k\in\begin{cases}(0,1/2)\\(1/2,\infty),\end{cases}\quad \sigma^{k,\pm k}_{\pm m} = \left[\frac{\Gamma(2k+m)}{\Gamma(2k)\Gamma(m+1)}\right]^{1/2} \leqslant 1, \text{(discrete)}.$$

(3.39c)

The inversion symmetry $k \leftrightarrow 1-k$ is not explicitly present in this formula, but a unitary intertwining operator may be set up to bridge these two representations (45, Sects. 2.3 and 2.4).

3.16 In search of Hilbert spaces.

We have now sets of functions $\phi_{\mu}^{k}(\phi)$ where $\mu_{\pm}$ is in the spectrum of $\mathbb{J}_0$ in a given, definite representation, (C_q^{ε} or $\mathcal{D}_k^{\pm}$) $\phi \in S_1$ (the circle) given by (3.8). We search for inner products where these are orthonormal and complete. Completeness of these denumerable bases, when proven, will define the Hilbert spaces as the closure of linear combinations of these functions.

The most general sesquilinear inner product on S_1 may be written as

$$(f,g)^{(k,\varepsilon)} = \int_{S_1} d\phi \int_{S_1} d\phi' \; f(\phi)^* \, \Omega^{(k,\varepsilon)}(\phi,\phi') \, g(\phi'). \qquad (3.40)$$

Under this inner product the algebra generators (3.35) must be hermitian: $(\mathbb{J}_j^{(k)} f,g)^{(k,\varepsilon)} = (f,\mathbb{J}_j^{(k)} g)^{(k,\varepsilon)}$. We have seen before the requirements imposed by $j=1,2$, i. e. by $J_+^{(k)\dagger} = J_-^{(k)}$. We now impose the hermiticity of $\mathbb{J}_0^{(k)}$, integrating by parts:

$$\begin{aligned}
(\mathbb{J}_0^{(k)} f,g)^{(k,\varepsilon)} &= \int_{S_1} d\phi \int_{S_1} d\phi' (-i\partial_\phi f(\phi))^* \, \Omega^{(k,\varepsilon)}(\phi,\phi') \, g(\phi') \\
&= \int_{S_1} d\phi \int_{S_1} d\phi' \; f(\phi)^* [-i\partial_\phi \Omega^{(k,\varepsilon)}(\phi,\phi')] g(\phi') \\
=(f,\mathbb{J}_0^{(k)} g)^{(k,\varepsilon)} &= \int_{S_1} d\phi \int_{S_1} d\phi' \; f(\phi)^* \, \Omega^{(k,\varepsilon)}(\phi,\phi')[-i\partial_{\phi'} g(\phi')] \\
&= \int_{S_1} d\phi \int_{S_1} d\phi' \; f(\phi)^* [i\partial_{\phi'} \Omega^{(k,\varepsilon)}(\phi,\phi')] g(\phi').
\end{aligned} \qquad (3.41)$$

This leads to

$$\int_{S_1} d\phi \int_{S_1} d\phi' \; f(\phi)^* [(\partial_\phi - \partial_{\phi'}) \Omega^{(k,\varepsilon)}(\phi,\phi')] g(\phi') = 0 \qquad (3.42)$$

which is to hold for all $f(\phi)$ and $g(\phi')$ in a space $\mathcal{H}^{(k,\varepsilon)}$. There is a rather subtle point to the ensuing reasoning, and that is since $\mathcal{H}^{(k,\varepsilon)}$ may be different from $L^2(S_1)$, what (3.42) actually implies is that $z(\phi,\phi') = (\partial_\phi - \partial_{\phi'}) \Omega^{(k,\varepsilon)}(\phi,\phi')$, as a function of ϕ, is orthogonal in the sense of $L^2(S_1)$ to $f(\phi)$, and similarly orthogonal to $g(\phi')$ in the same sense. If we were to put $z(\phi,\phi')$ as an inner product measure in place of $\Omega(\phi,\phi')$ in (3.40), the result would be zero for all f,g in $\mathcal{H}^{(k,\varepsilon)}$, and hence $z(\phi,\phi')$ is equivalent to the zero function itself. On the basis of this equivalence, we may replace $z(\phi,\phi')$ by zero and thus conclude

$$(\partial_\phi - \partial_{\phi'}) \Omega^{(k,\varepsilon)}(\phi,\phi') = 0 \Rightarrow \Omega^{(k,\varepsilon)}(\phi,\phi') = \Omega^{k,\varepsilon}(\phi-\phi'), \qquad (3.43a)$$

i. e. Ω is a function only of the angle difference $\phi-\phi'$. As the basis functions $e^{i\mu\phi}$ with μ as specified by the representation C_q^{ε} or $\mathcal{D}_k^{\pm}$, is assumed complete in $\mathcal{H}^{(k,\varepsilon)}$, we may expand $\Omega^{(k,\varepsilon)}(\psi)$ as

$$\Omega^{(k,\varepsilon)}(\psi) = \sum_m^{(k)} \omega_m^{k,\varepsilon} e^{i(\varepsilon\pm m)\psi} , \tag{3.43b}$$

with $\Sigma^{(k)}$ as in (3.36), and $\omega_m^{k,\varepsilon}$ as yet unknown and put it in (3.40) with the orthonormality condition imposed on the basis functions, using the Fourier series orthogonality results for the exponential functions. We thus develop

$$\begin{aligned}
\delta_{m,m'} &= (\phi^k_{\varepsilon\pm m}, \phi^k_{\varepsilon\pm m'})^{(k,\varepsilon)} = \int_{S_1} d\phi \int_{S_1} d\phi' [K_\varepsilon^k \, \eta_{\varepsilon\pm m}\sigma^{k\varepsilon}_{\pm m} \, e^{i(\varepsilon\pm m)\phi}]^* \times \\
&\times [\sum_{m''}^{(k)} \omega^{k\varepsilon}_{m''} \, e^{i(\varepsilon+m'')(\phi-\phi')}][K_\varepsilon^k \, \eta_{\varepsilon\pm m'}\sigma^{k\varepsilon}_{\pm m'} e^{i(\varepsilon\pm m')\phi'}] \\
&= |K_\varepsilon^k|^2 [\eta_{\varepsilon\pm m} \, \sigma^{k\varepsilon}_{\pm m}]^* [\eta_{\varepsilon\pm m'} \, \sigma^{k\varepsilon}_{\pm m'}] \sum_{m''}^{(k)} \Omega^{k\varepsilon}_{m''} \times \\
&\times [\int_{S_1} d\phi \, e^{i(n''\mp n)\phi}][\int_{S_1} d\phi' \, e^{i(\pm n'-n'')\phi'}] = \\
&= \delta_{mm'} (2\pi)^2 |K_\varepsilon^k|^2 |\sigma^{k\varepsilon}_{\pm m}|^2 \, \omega^{k\varepsilon}_{\pm m} .
\end{aligned} \tag{3.44a}$$

It follows that the proper coefficients for the weight function $\Omega^{k,\varepsilon}(\phi-\phi')$ are

$$\begin{aligned}
\omega^{k\varepsilon}_{\pm m} &= [2\pi |K_\varepsilon^k \, \sigma^{k\varepsilon}_{\pm m}|]^{-2} = \frac{1}{(2\pi|K_\varepsilon^k|)^2} \left| \frac{(1-k\pm\varepsilon)_m}{(k\pm\varepsilon)_m} \right| \\
&= \frac{1}{(2\pi|K_\varepsilon^k|)^2} \left| \frac{\Gamma(\varepsilon+1-k)}{\Gamma(\varepsilon+k)} \, \frac{\Gamma(\varepsilon+k\pm m)}{\Gamma(\varepsilon+1-k\pm m)} \right| \\
&= \frac{1}{(2\pi|K_\varepsilon^k|)^2} \left| \frac{\Gamma(k\pm\varepsilon)}{\Gamma(1-k\pm\varepsilon)} \, \frac{\Gamma(1-k\pm\varepsilon+m)}{\Gamma(k\pm\varepsilon+m)} \right| .
\end{aligned} \tag{3.44b}$$

3.17 Local and nonlocal weight functions.

Let us now examine the various representation series in order to give the description of the $\mathcal{H}^{(k,\varepsilon)}$ spaces.

For the continuous nonexceptional series C_q^{ε} we have $m \in Z$ and (3.39a), since $\sigma^{k\varepsilon}_{\pm m}$ is only a phase. Then

$$\Omega^{1/2+i\rho,\varepsilon}(\psi) = (2\pi|K_\varepsilon^k|)^{-2} \sum_{m\in Z} e^{i(m+\varepsilon)\psi} = e^{i\varepsilon\psi}[2\pi|K_\varepsilon^k|^2]^{-1}\delta(\psi) . \tag{3.45a}$$

We have hence

$$\mathcal{H}^{k=1/2+i\rho,\varepsilon} = L^2(S_1) \tag{3.45b}$$

and the inner product is the ordinary L^2-product on the circle.

For the continuous exceptional series,

$$\Omega^{k,\varepsilon}(\psi) = \frac{1}{(2\pi|K_\varepsilon^k|)^2}\left[\sum_{m=0}^{\infty}\omega_m^{k\varepsilon}\, e^{i(\varepsilon+m)\psi} + \sum_{m=1}^{\infty}\omega_{-m}^{k\varepsilon}\, e^{i(\varepsilon-m)\psi}\right]$$

$$= \frac{e^{i\varepsilon\psi}}{(2\pi|K_\varepsilon^k|)^2}\left[\sum_{m=0}^{\infty}\frac{(1-k+\varepsilon)_m(1)_m}{(k+\varepsilon)_m}\frac{e^{im\psi}}{m!} + \sum_{m=0}^{\infty}\frac{(1-k-\varepsilon)_m(1)_m}{(k-\varepsilon)_m}\frac{e^{-im\psi}}{m!} - 1\right]$$

$$= \frac{e^{i\varepsilon\psi}}{(2\pi|K_\varepsilon^k|)^2}\left[{}_2F_1(1,1-k+\varepsilon;k+\varepsilon;e^{i\psi}) + {}_2F_1(1,1-k-\varepsilon;k-\varepsilon;e^{-i\psi}) - 1\right]. \tag{3.46a}$$

When $\varepsilon=0$ it was proven by Bargmann (23, Eqs. (8.7), (8.9) and (8.11)) that this expression reduces to

$$\Omega^{k,0}(\psi) = \frac{1}{(2\pi|K_0^k|)^2}\sum_{m=-\infty}^{\infty}\frac{(1-k)_{|m|}}{(k)_{|m|}}\, e^{im\psi} = \frac{2^{-1-k}}{\pi\sqrt{\pi}|K_0^k|^2}\,\frac{\Gamma(k)}{\Gamma(k-1/2)}\,(1-\cos\psi)^{k-1}. \tag{3.46b}$$

Bargmann (23, Sect. 8c) also proves that $\mathcal{H}^{k,0}$ is a Hilbert space when $k\in(1/2,1)$, while Sally (45, Sect. 2.4) proves this for the remaining cases.

For the discrete series $\mathcal{D}_k^{\pm}$, finally, we can use (3.39c)-(3.44) in order to find

$$\begin{aligned}\Omega^{k,\pm k}(\psi) &= \sum_{m=0}^{\infty}\omega_{\pm m}^{k,\pm k}\, e^{\pm i(k+m)\psi} \\ &= \frac{e^{\pm ik\psi}}{(2\pi|K_{\pm k}^k|)^2}\;{}_2F_1(1,1;2k;e^{i\psi}).\end{aligned} \tag{3.47}$$

This series converges absolutely for $k \geqslant 1$, conditionally for $1/2 < k < 1$ (excluding $\psi=0,2\pi,\ldots$ although its integral with bounded functions is bounded), and diverges for $k=1/2$. For $0 < k < 1/2$ this series diverges but may still be used to define an inner product between appropiately convergent functions.

The nonlocal inner product (3.47) has been examined in (51) where it is proven that it defines a Hilbert space. There (37, Appendix) it is also proven that this representation is equivalent to the description of $\mathcal{D}_k^{\pm}$ by Bargmann and Sally (23, 45) as an integral over the complex unit disk, and by Gel'fand (51, Chapter VII) for single-valued representations of SO(2,1) on the upper complex half-plane. In the Hilbert space $\mathcal{H}^{k,\pm k}$, the operator $\mathbb{J}_0^{(k)} = -i\partial_\phi$ has thus a spectrum given by $\pm k, \pm(k+1), \pm(k+2), \ldots$, in contrast with the usual $L^2(S_1)$, where it is Z.

3.18 Unitary group action and multipliers.

The measures found above also make the exponentiated action of the algebra generators unitary. Let us make this explicit. The $\overline{SL(2,R)}$ action on functions belonging to $\mathcal{H}^{k,\varepsilon}$ is found from (3.29) and (3.34). It is convenient to parametrize the acting group through Euler angles (3.6). The action of $\exp(i\alpha\mathbb{J}_0)$ and $\exp(i\beta\mathbb{J}_2)$ on the coset parameters of $M=P\backslash SL(2,R)$ are found giving in (3.29) the g_0 parameters through their values (3.10), namely

$$e^{i\alpha\mathbb{J}_0^M} : \tan\frac{\phi}{2} \longrightarrow \tan\frac{\phi_0(\phi,\alpha)}{2} = \tan\frac{\phi+\alpha}{2} \qquad (3.48a)$$

$$e^{i\alpha\mathbb{J}_0^M} : e^{\chi} \longrightarrow e^{\chi_0(\chi,\phi,\alpha)} = e^{\chi} \qquad (3.48b)$$

$$e^{i\beta\mathbb{J}_2^M} : \tan\frac{\phi}{2} \longrightarrow \tan\frac{\phi_2(\phi,\beta)}{2} = e^{-\beta}\tan\frac{\phi}{2} \qquad (3.49a)$$

$$e^{i\beta\mathbb{J}_2^M} : e^{\chi} \longrightarrow e^{\chi_2(\chi,\phi,\beta)} = e^{\chi}[\,\mathrm{ch}\beta+\cos\frac{\phi}{2}\,\mathrm{sh}\beta] = e^{\chi}\frac{\sin\phi}{\sin\phi_2(\phi,\beta)}. \qquad (3.49b)$$

From here, it follows through (3.34) that the $\overline{SL(2,R)}$ group action on functions in the (k,ε)-irreducible representation spaces $\mathcal{H}^{k,\varepsilon}$ is

$$e^{-k\chi} f_k(\phi) \xrightarrow{e^{i\gamma\mathbb{J}_\bullet^M}} e^{-k\chi'(\chi,\phi,\gamma)} f_k(\phi'(\phi,\gamma)), \qquad (3.50a)$$

and hence, on functions on S_1,

$$f_k(\phi) \xrightarrow{e^{i\gamma\mathbb{J}_\bullet^{(k)}}} e^{-k\chi'(\chi,\phi,\gamma)}\, e^{k\chi} f_k(\phi'(\phi,\gamma))$$
$$= [\mu_\bullet(\phi,\gamma)]^k\, f_k(\phi'(\phi,\gamma)). \qquad (3.50b)$$

The group action has thus become a *multiplier* action. As in Chapter 2, the function's argument is transformed, and the function itself is multiplied by a factor

$$\mu_\bullet(\phi,\gamma) = e^{-\chi'(\chi,\phi,\gamma)}\, e^{\chi}. \qquad (3.50c)$$

For the two generating Euler subgroups of (3.48), we have

$$e^{i\alpha \mathbb{J}_0^{(k)}} \quad : \quad \mu_0(\phi,\alpha) = 1 \tag{3.51a}$$

$$e^{i\beta \mathbb{J}_2^{(k)}} \quad : \quad \mu_2(\phi,\beta) = \mathrm{ch}\beta + \cos\frac{\phi}{2}\,\mathrm{sh}\beta = \frac{\sin\phi}{\sin\phi_2(\phi,\beta)} = \left[\frac{\partial\phi_2(\phi,\beta)}{\partial\phi}\right]^{-1}.$$

Putting together (3.48a), (3.50b) and (3.51a) we see that the action of $exp(i\alpha\mathbb{J}_0)$ is a pure translation in S_1. This is a manifestly unitary transformation

$$(e^{i\alpha\mathbb{J}_0^{(k)}} f,\ e^{i\alpha\mathbb{J}_0^{(k)}} g)^{(k,\varepsilon)} = (f,\ g)^{(k,\varepsilon)}, \tag{3.52a}$$

in any of the $\mathcal{H}^{k\varepsilon}$ spaces seen above, as shown by a simple change of variables in the integral.

As to the action of $\exp(i\beta\mathbb{J}_2)$, we see that it *deforms* the circle parameter in the sense that $\partial\phi_2(\phi,\beta)/\partial\phi \neq 1$. Interestingly, as the last equality in (3.51b) shows, this is exactly given by the multiplier function. Now from (3.49a), (3.50b) and (3.51b), we may show that

$$(e^{i\beta\mathbb{J}_2^{(k)}} f,\ e^{i\beta\mathbb{J}_2^{(k)}} g)^{(k,\varepsilon)} = (f,\ g)^{(k,\varepsilon)}, \tag{3.52b}$$

for all the spaces $\mathcal{H}^{k,\varepsilon}$. This is clear for the nonexceptional continuous series, where in the $L^2(S_1)$ inner product, the change in $d\phi$ is $[\mu_2(\phi,\beta)]^{-1}$ which is exactly offset by the product of the multipliers: $([\mu_2(\phi,\beta)]^{1/2+i\rho})^* (\mu_2(\phi,\beta))^{1/2+i\rho}$. The element which is pleasing in this approach is that unitarity (3.52b) holds for all series when the appropiate inner product and weight function (3.46) and (3.47) are used.

Once appropiate Hilbert spaces have been found -and for SL(2,$\mathbb{R}$) we have seen there are three families of them- the irreducible representation matrix elements may be readily calculated.

3.19 Closing remarks.

Since we must stop at some point, we would like to insist on the fact that SL(2,$\mathbb{R}$), for all its 'simplicitly', is a rather richly structured object of which we have only given an overall view. Books devoted to SL(2,$\mathbb{R}$) exist (53). We should bear in mind that this is just one example of a noncompact group. There exist others in higher dimensions, nevertheless, the systematics in their treatment are not very different from what we have seen for their smallest representative member.

We shall resist the temptation to close thise lectures notes with a barrage of references, and simply point out that group theory, in spite of its simple and very compact four defining axioms, appears to contain a noncompact body of results★

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